Quantitative approximation of kinetic SDEs: from discrete to continuum

Zimo Hao

Joint work with Khoa Lê and Chengcheng Ling

Bielefeld University

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New trends of stochastic nonlinear systems: well-posedeness, dynamics and numerics

CIRM-Marseille

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Kinetic models

- Newton's Second Law of Motion:

$$\dot{V}_t = \ddot{X}_t = F(X_t, V_t)$$

- The distribution $\mu_t = \mu_t(\mathrm{d}x, \mathrm{d}v) := \mathbb{P} \circ (X_t, V_t)^{-1}$ solves the following kinetic equation:

$$\partial_t \mu_t + v \cdot \nabla_x \mu_t + \mathrm{div}_v(F \mu_t) = 0$$

Kinetic models

- Newton's Second Law of Motion:

$$\dot{V}_t = \ddot{X}_t = F(X_t, V_t) + \sqrt{2}\dot{B}_t.$$

- The distribution $\mu_t = \mu_t(\mathrm{d}x, \mathrm{d}v) := \mathbb{P} \circ (X_t, V_t)^{-1}$ solves the following kinetic equation:

$$\partial_t \mu_t + v \cdot \nabla_x \mu_t + \mathrm{div}_v(F\mu_t) = \Delta_v \mu_t.$$

- **Mesoscopic** (Boltzmann equation) and **macroscopic** (Navier-Stokes equations), ..., Villani, Lions, Golse, Bouchut, Imbert, Mouhot, Silvestre, Guo, Mourrat, ...
- **Microscopic** (Kinetic SDEs) and **mesoscopic** (Boltzmann equation): Tanaka 1978, PTRF., Mischler-Mouhot 2013, Invent.,...
- Difficulty: degenerate parabolic PDE.

Kinetic SDEs and Euler-Maruyama scheme

- Kinetic SDEs:

$$\begin{cases} \mathrm{d}X_t = V_t \mathrm{d}t, \\ \mathrm{d}V_t = b(X_t, V_t)\mathrm{d}t + \mathrm{d}B_t. \end{cases}$$

- Let $n \in \mathbb{N}$ and $k_n(t) := [nt]/n$. Euler-Maruyama scheme:

$$\begin{cases} \mathrm{d}X^n_t = V^n_t \mathrm{d}t, \\ \mathrm{d}V^n_t = b(X^n_{k_n(t)}, V^n_{k_n(t)})\mathrm{d}t + \mathrm{d}B_t. \end{cases}$$

Euler-Maruyama scheme

- Let $n \in \mathbb{N}$ and $k_n(t) := [nt]/n$. Euler-Maruyama scheme:

$$\begin{cases} \mathrm{d}X_t^n = V_t^n \mathrm{d}t, \\ \mathrm{d}V_t^n = b(X_{k_n(t)}^n, V_{k_n(t)}^n)\mathrm{d}t + \mathrm{d}B_t. \end{cases}$$

- Let $h := 1/n$. For $s \in [kh, (k+1)h]$,

$$X_{(k+1)h}^n = X_{kh}^n + \int_{kh}^{(k+1)h} V_s^n \mathrm{d}s,$$
$$V_s^n = V_{kh}^n + (s-kh)b(X_{kh}^n, V_{kh}^n) + W_s - W_{kh}.$$

- Recursive relation:

$$\begin{pmatrix} X_{(k+1)h}^n \\ V_{(k+1)h}^n \end{pmatrix} = F_h^n(X_{kh}^n, V_{kh}^n) + \xi_k^n, \tag{1}$$

where $k = 0, 1, 2, \ldots,$

$$F_h^n(x,v) := \begin{pmatrix} x + hv + \frac{h^2}{2} b(r,x,v) \\ v + hb(x,v) \end{pmatrix}, \quad \xi_k^n = \begin{pmatrix} \int_{kh}^{(k+1)h}(W_s - W_{kh})\mathrm{d}s \\ W_{(k+1)h} - W_{kh} \end{pmatrix}.$$

- ξ_k^n are i.i.d. Gaussian random variables with covariance matrix

$$\begin{pmatrix} \frac{h^3}{3}\mathbb{I}_{d\times d} & \frac{h^2}{2}\mathbb{I}_{d\times d} \\ \frac{h^2}{2}\mathbb{I}_{d\times d} & h\mathbb{I}_{d\times d} \end{pmatrix}.$$

Selected results

- (Lemaire-Menozzi, 2010. EJP) b is bounded:
  Weak convergence: $\mathbb{P} \circ (X^n, V^n)^{-1} \to \mathbb{P} \circ (X, V)^{-1}$, as $n \to \infty$.
  (Heat kernel estimates)

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  (Heat kernel estimates)
- (Leobacher-Szölgyenyi, 2018. Numer. Math.)
  b is piecewise continuous and bounded:

$$\mathbb{E}\left(\sup_{t\in[0,1]} |(X_t, V_t) - (X_t^n, V_t^n)| \right) \lesssim n^{-\frac{1}{4}+}.$$

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- (Question)
  - What about the singular case like

$$b(x, v) = x/|x|^{-1-\gamma}, \quad \gamma > 0?$$

  - Can the convergence rate be improved?
- Let $p_x, p_v \in [1, \infty]$. Define the mix L^p space with the norm:

$$\|b\|_{L_x^{p_x} L_v^{p_v}} := \left(\int_{\mathbb{R}_v^d} \left|\int_{\mathbb{R}_x^d} |b(x, v)|^{p_x} \mathrm{d}x\right|^{\frac{p_v}{p_x}} \mathrm{d}v\right)^{\frac{1}{p_v}}.$$

Kinetic semi-group and transport

- Let $b = 0$.

$$(X_t, V_t) = \left(\int_0^t B_s \mathrm{d}s, B_t \right).$$

- Heat kernel estimate: (Kolmogorov, 1934): $p_t(x, v) \sim (X_t, V_t)$,

$$p_t(x, v) = (4\pi t^4/3)^{-d/2} \exp\left(-(3|x|^2 + |3x - 2tv|^2)/(4t^3) \right).$$

  - (Chaudru-Menozzi-Pesce-Zhang, 2023. Bull. Sci. Math.), (Ren-Zhang, 2025. Bernoulli) ...

- Kinetic semi-group:

$$P_t f(x, v) = \mathbb{E} f(x + tv + X_t, v + V_t) = \Gamma_t (p_t * f)(x, v) = (\Gamma_t p_t) * (\Gamma_t f)(x, v),$$

where the transport operator Γ_t is defined by

$$\Gamma_t f(x, v) := f(x + tv, v).$$

- $\partial_t P_t f = (\frac{1}{2}\Delta_v + v \cdot \nabla_x) P_t f.$

Kinetic scaling

- Recall kinetic SDEs:

$$\begin{cases} \mathrm{d}X_t = V_t \mathrm{d}t, \\ \mathrm{d}V_t = b(X_t, V_t)\mathrm{d}t + \mathrm{d}B_t. \end{cases}$$

- Scaling ($b = 0$):

$$\begin{aligned} (X_{\varepsilon^2 t}, V_{\varepsilon^2 t}) &= \Big(\int_0^{\varepsilon^2 t} B_s \mathrm{d}s, B_{\varepsilon^2 t}\Big) = \Big(\int_0^t \varepsilon^2 B_{\varepsilon^2 s'} \mathrm{d}s', B_{\varepsilon^2 t}\Big) \\ &\overset{(d)}{=} \Big(\varepsilon^3 \int_0^t B_{s'} \mathrm{d}s', \varepsilon B_t\Big) = (\varepsilon^3 X_t, \varepsilon V_t). \end{aligned}$$

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$$\overset{(d)}{=} \left(\varepsilon^3 \int_0^t B_{s'} \mathrm{d}s', \varepsilon B_t\right) = (\varepsilon^3 X_t, \varepsilon V_t).$$

- Anisotropic scaling: $X : V = 3:1$.
- $$p_t(x, v) = t^{-2d} p_1(t^{-3/2} x, t^{-1/2} v).$$

Well-posedness

- Kinetic SDEs: $V = \dot{X}$,

$$\mathrm{d}\dot{X}_t = b(X_t, \dot{X}_t)\mathrm{d}t + \mathrm{d}B_t.$$

- Weak well-posedness [Weak existence + Uniqueness in law]
  - (Chaudru de Raynal-Menozzi, 2021. TAMS) $b \in L^p_{x,v}$, $\boxed{\frac{4d}{p} < 1}$.
  - (Ren-Zhang, 2025. Bernoulli) Kato class, $b \in L^{p_x}_x L^{p_v}_v$, $\boxed{\frac{3d}{p_x} + \frac{d}{p_v} < 1}$.

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- Strong well-posedness [Strong existence + Pathwise uniqueness]
  - (Chaudru de Raynal, 2017. AIHP) $b \in C_x^{\frac{2}{3}+} \cap C_v^{0+}$.
  - (Wang-Zhang, 2016. SIAM) Dini continuous.
  - (Fedrizzi-Flandoli-Priola-Vovelle, 2017. EJP) $(1-\Delta_x)^{\frac{1}{3}+} b \in L^p_{x,v}$, $p > 6d$.
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- Distribution drift: (H.-Zhang-Zhu-Zhu, 2024. AOP), (Issoglio-Pagliarani-Russo-Trevisani, 2024.), (Chen-H.-Zhang, 2025) ...

Our aim

- Let $p_x, p_v \in [2, \infty]$ with

$$\frac{3d}{p_x} + \frac{d}{p_v} < 1.$$

	$b \in L_x^{p_x} L_v^{p_v}$	$(1 - \Delta_x)^{\frac{1}{3}} b \in L_x^{p_x} L_v^{p_v}$
Well-known:	Weak well-posedness	Strong well-posedness if $p_x = p_v > 6d$
Our aim:	Weak convergence	Strong well-posedness & Strong convergence

Tamed Euler-Maruyama scheme with time transport

- Recall $\boxed{\Gamma_t f(x,v) := f(x+tv,v)}$ and let $\boxed{b_n := b * \phi_n}$, where for a smooth probability density function ϕ and $\theta > 0$,

$$\phi_n(x,v) := n^{4d\theta}\phi(n^{3\theta}x, n^{\theta}v).$$

- Tamed Euler-Maruyama scheme:

$$\begin{cases} \mathrm{d}X_t^n = V_t^n \mathrm{d}t, \\ \mathrm{d}V_t^n = \Gamma_{t-k_n(t)} b_n(X_{k_n(t)}^n, V_{k_n(t)}^n)\mathrm{d}t + \mathrm{d}B_t. \end{cases}$$

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- Benefits of adding $\Gamma_{t-k_n(t)}$:
  Let $X_t := \int_0^t B_s ds$ and $f(x,v) = f(x)$ be a Lipschitz function.
  - (Without $\Gamma_{t-k_n(t)}$) : $\mathbb{E}|f(X_t) - f(X_{k_n(t)})| \leq \|f\|_{Lip} \int_{k_n(t)}^t s^{1/2} ds \lesssim n^{-1}$.

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  - (With $\Gamma_{t-k_n(t)}$) :

$$\begin{aligned} \mathbb{E}|f(X_t) - \Gamma_{t-k_n(t)} f(X_{k_n(t)})| &\leq \|f\|_{Lip} \int_{k_n(t)}^t \mathbb{E}|W_s - W_{k_n(s)}|\mathrm{d}s \\ &\lesssim \int_{k_n(t)}^t (s - k_n(s))^{1/2}\mathrm{d}s \lesssim n^{-3/2}. \end{aligned}$$

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- $P_t f - f$ is replaced with $P_t f - \Gamma_t f$ (Imbert-Silvestre, 2021. Anal. PDE), (H.-Zhang-Zhu-Zhu, 2024. AOP), (Zhang-Zhang, 2024. AAP), ...

Weak convergence

Main results

- Assume that $b \in L_x^{p_x} L_v^{p_v}$ with

$$\alpha := \frac{3d}{p_x} + \frac{d}{p_v} < 1.$$

Theorem 1

Let $\theta \in (0, \frac{1}{2\alpha}]$. *For any* $T > 0$,

$$\sup_{t \in [0,T]} t^{(\theta \vee \frac{1}{2})\alpha} \|\mathbb{P} \circ (X_t, V_t)^{-1} - \mathbb{P} \circ (X_t^n, V_t^n)^{-1}\|_{\text{var}} \lesssim n^{-\frac{1}{2}} + n^{-\theta}.$$

- When b is bounded, we can replace b_n with b and have

$$\sup_{t \in [0,T]} \|\mathbb{P} \circ (X_t, V_t)^{-1} - \mathbb{P} \circ (X_t^n, V_t^n)^{-1}\|_{\text{var}} \lesssim n^{-\frac{1}{2}}.$$

Remarks:

(i) We don't need any continuous assumption on b.

(ii) When $b = b(v)$, the kinetic SDE reduces to

$$\mathrm{d}V_t = b(V_t)\mathrm{d}t + \mathrm{d}B_t. \tag{2}$$

We can replace the taming $b_n := b * \phi_n$ with

$$b_n(v) := \frac{|b(v)| \wedge (Cn^{\alpha\theta})}{|b(v)|} b(v)$$

for some constant $C > 0$.

In (Jourdain-Menozzi, 2024. AAP), where $b \in L^p$ with $p > d$ and $\theta = 1/(2\alpha)$, they obtained the convergence of the density, which in particular implies

$$\|\mathbb{P} \circ (V_t)^{-1} - \mathbb{P} \circ (V_t^n)^{-1}\|_{\mathrm{var}} \lesssim n^{-\frac{1-d/p}{2}}.$$

In contrast, our weak convergence rate is $1/2$, which is independent of p.

Strong convergence

Main results

- Assume that $b \in \mathbf{B}^{\beta}_{(p_x,p_v);a}$ (anisotropic Besov space) with $\beta \in (0,1)$,

$$\gamma := -\beta + \frac{3d}{p_x} + \frac{d}{p_v} < 1,$$

and assume

$$(1-\Delta_x)^{\frac{1}{3}} b \in \cup_{\varepsilon>0} \mathbf{B}^{\varepsilon}_{(p_x,p_v);a}. \tag{3}$$

- $\mathbf{B}^{\beta}_{(p_x,p_v);a} \approx \left[(1-\Delta_x)^{-\frac{\beta}{6}} L^{p_x}_x L^{p_v}_v\right] \cap \left[(1-\Delta_v)^{-\frac{\beta}{2}} L^{p_x}_x L^{p_v}_v\right].$

Theorem 2

- *Under the assumption* (3) *with* $\alpha < 1$*, the strong well-posedness holds.*
- *If* $\gamma \geq 0$*, then for any* $\theta \in (0, \frac{1}{2\gamma}]$, $\varepsilon > 0$, $T > 0$ *and* $m \geq 1$,

$$\left(\mathbb{E}\left[\sup_{t\in[0,T]} |(X_t,V_t) - (X^n_t, V^n_t)|^m\right]\right)^{\frac{1}{m}} \lesssim n^{-\frac{1+\beta}{2}+\varepsilon} + n^{-(1-\gamma)\theta+\varepsilon}.$$

- *If* $\gamma < 0$*, then* $b \in L^\infty_{x,v}$*, we can replace the taming* b_n *with* b *and have*

$$\left(\mathbb{E}\left[\sup_{t\in[0,T]} |(X_t,V_t) - (X^n_t, V^n_t)|^m\right]\right)^{\frac{1}{m}} \lesssim n^{-\frac{1+\beta}{2}+\varepsilon}.$$

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- If $\gamma \geq 0$, then for any $\theta \in (0, \frac{1}{2\gamma}]$, $\varepsilon > 0$, $T > 0$, and $m \geq 1$

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Remarks:

(i) Strong well-posedness holds without assuming $p_x = p_v$, covering both cases $b = b(x)$ and $b = b(v)$.

(i) The strong convergence rate exceeds $1/2$ when

$$\gamma < 1/2 \quad \text{and} \quad \theta \in (1/(2(1-\gamma)), 1/(2\gamma)].$$

(ii) When $b = b(v)$, our results coincide with those in (Dareiotis-Gerencsér-Lê, 2023. AAP, Lê-Ling, 2025. AOP) for non-degenerate SDE.

(iv) When $b = b(x)$, the condition becomes to $b \in \mathbf{B}_{p_x}^{\frac{\beta}{3}}$. In other words, we achieve a convergence rate of $\frac{1+\beta}{2}$ assuming that b has regularity $\frac{\beta}{3}$.

Main results

Theorem 3

Assume $b = b(x) \in \mathbf{C}^{\frac{\beta}{3}}$ *with some* $\beta \in (2,3)$. *Then we can replace* b_n *with* b *and have*

$$\left(\mathbb{E}\left[\sup_{t\in[0,T]} |(X_t, V_t) - (X_t^n, V_t^n)|^m\right]\right)^{\frac{1}{m}} \lesssim n^{-\frac{1+\beta}{2}+\varepsilon}.$$

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- **Strong convergence:** The strong convergence exceeds $3/2$ and approaches 2 as $\gamma \to 1$.
- **Reason:** This is due to the second-order construction and our scheme:

$$\begin{aligned} \mathrm{d}V_t^n &= \Gamma_{t-k_n(t)} b(X_{k_n(t)}^n, V_{k_n(t)}^n)\mathrm{d}t + \mathrm{d}B_t \\ &= b(X_{k_n(t)}^n - (t - k_n(t))\dot{X}_{k_n(t)}^n)\mathrm{d}t + \mathrm{d}B_t. \end{aligned}$$

- **Relation to the Milstein scheme:**

$$\mathrm{d}V_t^n = \left[b(X_{k_n(t)}^n) - (t - k_n(t))V_{k_n(t)}^n \cdot \nabla b(X_{k_n(t)}^n)\right] \mathrm{d}t + \mathrm{d}B_t;$$

- **Related work:** A similar scheme was studied in Vanden-Eijnden and Ciccotti (2006, Chemical Physics Letters) for very smooth b:

$$\mathrm{d}V_t^n = \frac{1}{2}\left[b(X_{k_n(t)}^n) + b(X_{k_n(t)+\frac{1}{n}}^n)\right] \mathrm{d}t + \mathrm{d}B_t.$$

Purely discrete scheme

- Assume b is bounded. Consider the purely discrete scheme:

$$\begin{cases} X_t^n = \xi + \int_0^t V_s^n \mathrm{d}s, \\ V_t^n = \eta + \int_0^t \Gamma_{k_{n^2}(s)-k_n(s)} b(X_{k_n(s)}^n, V_{k_n(s)}^n)\mathrm{d}s + B_t. \end{cases}$$

- Weak convergence:

$$\sup_{t\in[0,T]} \left\| \mathbb{P}\circ(X_t^n, V_t^n)^{-1} - \mathbb{P}\circ(X_t, V_t)^{-1} \right\|_{\mathrm{var}} \lesssim n^{-\frac{1}{2}}.$$

- Strong convergence: assume that $b \in \mathbf{C}_x^{\frac{\beta}{3}} \cap \mathbf{C}_v^{\beta}$ with $\beta \in (0,1)$ and $b \in \mathbf{C}_x^{\frac{2}{3}+}$, then for any $\varepsilon > 0$,

$$\left\| \sup_{t\in[0,T]} |(X_t, V_t) - (X_t^n, V_t^n)| \right\|_{L^m(\Omega)} \lesssim n^{-\frac{1+\beta}{2}+\epsilon}.$$

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$$\sup_{t\in[0,T]} \left\| \mathbb{P}\circ (X_t^n, V_t^n)^{-1} - \mathbb{P}\circ (X_t, V_t)^{-1} \right\|_{\text{var}} \lesssim n^{-\frac{1}{2}}.$$

- Strong convergence: assume that $b \in \mathbf{C}_x^{\frac{\beta}{3}} \cap \mathbf{C}_v^{\beta}$ with $\beta \in (0,1)$ and $b \in \mathbf{C}_x^{\frac{2}{3}+}$, then for any $\varepsilon > 0$,

$$\left\| \sup_{t\in[0,T]} |(X_t, V_t) - (X_t^n, V_t^n)| \right\|_{L^m(\Omega)} \lesssim n^{-\frac{1+\beta}{2}+\epsilon}.$$

Thank you for your time and attention!

Merci beaucoup!