Two works about Euler approximation for SDEs

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  - **Introduction**
  - **Main results**

Part 1 : Euler approximation for DDSDEs of Neymytskii-type

Based on the joint work[1] with Michael Röckner and Xicheng Zhang

[1] Hao, Z., Röckner, M. and Zhang, X., Euler scheme for density dependent stochastic differential equations. arXiv:2007.15426.

DDSDE

- Let $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t\geqslant 0})$ be a complete filtration probability space.
- Let $(W_t)_{t\geqslant 0}$ be a d-dimensional standard $\mathscr{F}_t$-Brownian motion.
- Denote by $\mathcal{P}(\mathbb{R}^d)$ the space of all probability measures over $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, which is endowed with the weak convergence topology.
- Let $b : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P} \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P} \to \mathbb{R}^d \otimes \mathbb{R}^d$ be two measurable functions.
- Consider the following distribution dependent stochastic differential equation (abbreviated as DDSDE):

$$\mathrm{d}X_t = b(t, X_t, \mu_t)\mathrm{d}t + \sigma(t, X_t, \mu_t)\mathrm{d}W_t, \quad X_0 \overset{(d)}{=} \mu_0, \tag{1.1}$$

where $\mu_t = \mathbb{P} \circ X_t^{-1}$.

DDSDE and NFPE

- By Itô's formula, under some general conditions on the coefficients, for any $\varphi \in \mathbf{C}_b^\infty(\mathbb{R}^d)$, μ_t satisfies the following nonlinear Fokker-Planck equation (abbreviated as NFPE):

$$\int_{\mathbb{R}^d} \varphi(x)\mu_t(\mathrm{d}x) = \int_{\mathbb{R}^d} \varphi(x)\mu_0(\mathrm{d}x) + \int_0^t \int_{\mathbb{R}^d} (L_{\mu_s}\varphi)(s,x)\mu_s(\mathrm{d}x)\mathrm{d}s, \tag{1.2}$$

where for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $a_{ij} := \sum_{k=1}^d \sigma_{ik}\sigma_{jk}$,

$$(L_{\mu_t}\varphi)(t,x) := \frac{1}{2}\sum_{i,j=1}^d a_{i,j}(t,x,\mu_t)\partial_i\partial_j\varphi(x) + \sum_{i=1}^d b_i(t,x,\mu_t)\partial_i\varphi(x).$$

DDSDE and NFPE

- Existence of DDSDE ⇒ Existence of NFPE;
- Assume that
  - *(i)* $\mu_t \in \mathcal{P}(\mathbb{R}^d)$ for all $t \in \mathbb{R}_+$.
  - *(ii)* $\forall i, j = 1, .., d,$

$$\int_0^T \int_{\mathbb{R}^d} [|a_{ij}(t, x, \mu_t)| + |b_i(t, x, \mu_t)|] \mu_t(\mathrm{d}x)\mathrm{d}t < \infty \quad \forall T > 0.$$

  - *(iii)* $t \to \mu_t$ is weakly continuous.

  By the superposition principle (see Section 2 in [1] and Theorem 2.5 in [2]), we have
- Existence of NFPE ⇒ Existence of DDSDE;
- Weak uniqueness of DDSDE ⇒ Uniqueness of NFPE.

[1] Barbu, V., Röckner, M., From Fokker-Planck equations to solutions of distribution dependent SDE, to appear in Annals of Probability. https://doi.org/10.1214/19-AOP1410.

[2] Trevisan, D. Well-posedness of multidimensional diffusion processes with weakly differentiable coefficients. Electron. J. Probab. https://doi.org/10.1214/16-EJP4453.

Nemytskii-type

- In the special case, a, b only works on measures with density respect to the Lebesgue measure $\mathrm{d}x$ and there are $\bar{b} : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ and $\bar{\sigma} : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \otimes \mathbb{R}^d$ such that

$$\mathrm{d}X_t = \bar{b}(t, X_t, \rho_t(X_t))\mathrm{d}t + \bar{\sigma}(t, X_t, \rho_t(X_t))\mathrm{d}W_t, \quad X_0 \stackrel{(d)}{=} \mu_0,$$

  where $\rho_t(x) := \frac{\mathrm{d}\mu_t}{\mathrm{d}x}(x)$, which is called the Nemytskii-type.
- This time, NFPE can be rewritten (in the sense of Schwartz distributions) as

$$\partial_t \rho_t(x) = \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j [\bar{a}_{ij}(t, x, \rho_t(x))\rho_t(x)] - \mathrm{div}[\bar{b}(t, x, \rho_t(x))\rho_t(x)],$$

$$\lim_{t \downarrow 0} \rho_t = \nu_0 \text{ weakly},$$

  where $\bar{a}_{ij} = \sum_{k=1}^{d} \bar{\sigma}_{ik}\bar{\sigma}_{jk}$, which is a quasilinear parabolic equation.
- In the sequel, we only consider the DDSDE of Nemytskii-type. For simplicity, denote by σ, a, b the $\bar{\sigma}, \bar{a}, \bar{b}$.

Our model

- For simplicity, we consider an easy DDSDE of Nemytskii-type:

$$dX_t = b(t, X_t, \rho_t(X_t))dt + \sqrt{2}dW_t, \quad X_0 \overset{(d)}{=} \mu_0, \qquad (1.3)$$

and NFPE:

$$\partial_t \rho_t(x) = \Delta \rho_t(x) - \mathrm{div}[b(t, x, \rho_t(x))\rho_t(x)], \quad \lim_{t\downarrow 0} \rho_t = \mu_0 \text{ weakly.} \qquad (1.4)$$

Definition 1

Let $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$. We call a filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t\geqslant 0})$ together with a pair of processes (X, W) thereon a weak solution of SDE (1.3) with initial distribution μ_0, if

- $\mathbb{P} \circ X_0^{-1} = \mu_0$ and W is a d-dimensional $\mathscr{F}_t$-BM;
- for each $t > 0$, $\rho_t(x) := \frac{\mathbb{P}\circ X_t^{-1}(dx)}{dx}(x)$ and

$$X_t = X_0 + \int_0^t b(s, X_s, \rho_s(X_s))ds + \sqrt{2}W_t, \quad \mathbb{P} - a.s.$$

Question: In what conditions of b, the existence and uniqueness hold?

Known results

2018 (Barbu and Röckner, *Ann. Probab. 48(2020)*)

- Assume that μ_0 has a density with respect to the Lebesgue measure, $b(t,x,u) = b(x,u)$ and one of the followings holds:
  - *(i)* $b \in \mathbf{C}_b(\mathbb{R}^d \times \mathbb{R}) \cap \mathbf{C}^1(\mathbb{R}^d \times \mathbb{R})$, $b(x,0) \equiv 0, \forall x \in \mathbb{R}^d$;
  - *(ii)* $b \in \mathbf{C}_b(\mathbb{R}) \cap \mathbf{C}^1(\mathbb{R}^d)$, $b(0) = 0$.

  Then there exists a weak solution to DDSDE (1.3).

2019 (Barbu and Röckner, *arXiv:1909.04464*)

- Assume that μ_0 has a density $\rho_0(x)$ with respect to the Lebesgue measure, $b(t,x,u) = b(x,u)$, $b \in \mathbf{C}_b(\mathbb{R}^d \times \mathbb{R}) \cap \mathbf{C}^1(\mathbb{R}^d \times \mathbb{R})$, $b(x,0) \equiv 0$

$$\sup\{|\partial_r b^i(x,r)|; x \in \mathbb{R}^d, i = 1,2, |r| \leqslant M\} \leqslant C_M, \quad \forall M > 0,$$

  and, for

$$\delta(r) := \sup |\partial_x b(x,r)|; x \in \mathbb{R}^d,$$

  we have $\delta \in \mathbf{C}_b(\mathbb{R})$. For each $\rho_0 \in L^\infty \cap L^1$, the NFPE (1.4) has at most one distributional solution $\rho \in L^\infty(\mathbb{R}_+; L^1) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$.

- Actually, in the papers above, they mainly concentrate on the case $a_{i,j} \neq \delta_{i,j}$ with some assumptions on a. For simplicity, we assume $a_{i,j} = \delta_{i,j}$ and only show the assumption of b.

Known results

- Other results about DDSDE of Nemytskii-type:
- Barbu, V., Röckner, M., Probabilistic representation for solutions to nonlinear Fokker-Planck equations, SIAM J. Math. Anal., 50 (2018), 4246-4260.
- Barbu, V. and Röckner, M., Solutions for nonlinear Fokker-Planck equations with measures as initial data and McKean-Vlasov equations. arXiv:2005.02311.
- ......

- In all the above works, they obtained the results by solving the associated NFPE and then by the superposition principle.

New Question: Is it possible to use a purely probabilistic method to construct a weak solution?

- In fact, we shall use Euler's scheme to construct a weak solution.

Euler scheme

- Let $T > 0$, $N \in \mathbb{N}$ and $h := T/N$. For $t \in [0, h)$, define

$$X_t^N := X_0 + \sqrt{2}W_t.$$

- For $t \in [kh, (k+1)h)$, we inductively define X_t^N by

$$X_t^N := X_{kh}^N + (t - kh)b(kh, X_{kh}^N, \rho_{kh}^N(X_{kh}^N)) + \sqrt{2}(W_t - W_{kh}),$$

where $\rho_{kh}^N(x)$ is the distributional density of X_{kh}^N.
- All in all, X_t^N solved the following Euler scheme:

$$X_t^N = X_0 + \int_0^t b^N(\phi_N(s), X_{\phi_N(s)}^N)\mathrm{d}s + \sqrt{2}W_t,$$

where $\phi_N(s) := jh$ for $s \in [jh, (j+1)h)$ and

$$b^N(t, x) = 1_{t \geqslant h} b(t, x, \rho_{\phi_N(s)}^N(x)).$$

Main results

Theorem 2

Assume that b is bounded measurable and

$$\lim_{t\to t_0}\lim_{u\to u_0}\sup_{|x|<R}|b(t,x,u)-b(t_0,x,u_0)|=0,\quad \forall R>0. \tag{1.5}$$

(Existence) For any $T>0$ and initial data $\mu_0\in\mathcal{P}(\mathbb{R}^d)$, there are a subsequence N_k and a weak solution X_t to DDSDE (1.3), so that for any bounded measurable f and $t\in(0,T]$,

$$\lim_{k\to\infty}\mathbb{E}f(X_t^{N_k})=\mathbb{E}f(X_t). \tag{1.6}$$

Moreover, X_t admits a density ρ_t with

$$\lim_{k\to\infty}\int_{\mathbb{R}^d}|\rho_t^{N_k}(x)-\rho_t(x)|\mathrm{d}x=0. \tag{1.7}$$

(Uniqueness) Assume that $\mu_0(\mathrm{d}x)=\rho_0(x)\mathrm{d}x$ with $\rho_0\in L^1\cap L^q$ for some $q\in(d,\infty]$, and there is a constant C such that for all t,x,u_1,u_2,

$$|b(t,x,u_1)-b(t,x,u_2)|\leqslant C|u_1-u_2|. \tag{1.8}$$

Then weak and strong uniqueness hold for DDSDE (1.3).

Remark

- We emphasize that the continuity of b in the time variable is no necessary for the existence of weak solution. Here we need it because we are considering the Euler scheme.
- If the uniqueness holds, then limit (1.6) and (1.7) hold for the whole sequence.
- By the well-known results about heat kernel estimate, there are constants $C > 0$ and $\lambda \geqslant 1$ such that for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$\rho_t(x) \leqslant Ct^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{\lambda t}} \mu_0(\mathrm{d}y).$$

- Rewrite

$$b(t, x, \mu) = \bar{b}(t, x, \rho(x)),$$

where $\rho(x) := \frac{\mathrm{d}\mu}{\mathrm{d}x}(x)$. Notice that we can't compare the condition $b(t, x, \cdot)$ is continuous in $\mathcal{P}(\mathbb{R}^d)$ and the condition $\bar{b}(t, x, \cdot)$ is continuous in $\mathbb{R}$.

Main reults

Corollary 3

Let $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$.

(i) *Assume* b *is bounded and measurable such that* (1.5) *holds. Then there is a weak solution* ρ_t *to NFPE* (1.4) *in the distribution dense with* $\int_{\mathbb{R}^d} \rho_t(x)\mathrm{d}x = 1$ *and*

$$0 \leqslant \rho_t(x) \leqslant Ct^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{\lambda t}} \mu_0(\mathrm{d}y).$$

(ii) *Assume that* (1.8) *holds and that* $\mu_0(\mathrm{d}x) = \rho_0(x)\mathrm{d}x$ *with* $\rho_0 \in (L^1 \cap L^q)(\mathbb{R}^d)$ *for some* $q \in (d, \infty]$. *Then the solution in assertion (i) is unique.*

Existence

- Recall

$$X_t^N = X_0 + \int_0^t b^N(\phi_N(s), X^N_{\phi_N(s)})\mathrm{d}s + \sqrt{2}W_t, \qquad (1.9)$$

where $\phi_N(s) := jh$ for $s \in [jh, (j+1)h)$ and

$$b^N(t,x) = 1_{t\geqslant h} b(t, x, \rho^N_{\phi_N(s)}(x)).$$

- Firstly, we have

$$\mathbb{E}|X_t^N - X_s^N|^{2p} \leqslant C_p|t-s|^p,$$

for some unimportant C_p which is independent with N. By Kolmogorov's criterion, Prokhorov's theorem and Skorokhod's representation theorem, the law of X^N is tight and there is a new probability space with $(\tilde{X}, \tilde{W})$ and $(\tilde{X}^N, \tilde{W}^N)$ thereon which has the same distribution as (X^N, W) such that

$$(\tilde{X}^{N_k}, \tilde{W}^{N_k}) \to (\tilde{X}, \tilde{W}), \quad a.s.$$

for some subsequence N_k. For simplicity, we denote N_k by N.

- It is easy to see that W^N and W are BMs and $\tilde{X}^N$ satisfies the Euler scheme (1.9) with $W = \tilde{W}^N$.

Existence

- On the other hand, we shall obtain some properties of ρ_t^N.
- When $X_0 = x$, we denote by $X_t^N(x) := X_t^N$. Let

$$g(t,x) := \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}.$$

Lemma 4 (Duhamel's formula)

For each $t \in (0,T]$ and $x \in \mathbb{R}^d$, $X_t^N(x)$ admits a density $p_x^N(t,y)$ which satisfies the following equality:

$$p_x^N(t,y) = g(t,x-y) + \int_0^t \mathbb{E}\Big[b^N(\phi_N(s), X_{\phi_N(s)}^N)\nabla g(t-s, y-X_s^N)\Big]\mathrm{d}s.$$

Moreover, $\rho_t^N(y) = \int_{\mathbb{R}^d} p_x^N(t,y)\mu_0(\mathrm{d}x)$.

Theorem 5 (Lemaire-Menozzi(2010), EJP)

For any $T>0$, there is a constant $C = C(d, \|b\|_\infty, T)$ such that for all $N \in \mathbb{N}$, $t \in (0,T]$ and $x, y \in \mathbb{R}^d$,

$$p_x^N(t,y) \leqslant Cg(4t, x-y).$$

Existence

- By these two results, it is easy to see that ρ^N is uniformly bounded and Hölder in $[1/M, T] \times \mathbb{R}^d$ for any $M > 1$. Therefore, by Ascolli-Arzela's theorem, there is a function $\rho_t(x)$ and subsequence $\{N_k\}_k$ with

$$\lim_{k\to\infty} \sup_{t\in[1/M,T]} \sup_{|x|\leqslant M} |\rho_t^{N_k}(x) - \rho_t(x)| = 0, \quad \forall M > 0. \qquad (1.10)$$

Moreover, ρ_t is the density of $\tilde{X}_t$. For simplicity, denote N by N_k.

- $\tilde{X}_t^N = \tilde{X}_0 + \int_h^t b(\phi_N(s), \tilde{X}^N_{\phi_N(s)}, \rho^N_{\phi_N(s)}(\tilde{X}^N_{\phi_N(s)}))\mathrm{d}s + \sqrt{2}\tilde{W}_t^N$

$\downarrow \quad \downarrow \quad \downarrow$ (1.5) & (1.10) $\quad \downarrow$

- $\tilde{X}_t = \tilde{X}_0 + \int_h^t b(s, \tilde{X}^N_{\phi_N(s)}, \rho_s(\tilde{X}^N_{\phi_N(s)}))\mathrm{d}s + \sqrt{2}\tilde{W}_t$

$\downarrow \quad \downarrow \quad \downarrow$ mollify b & Krylov's estimate $\quad \downarrow$

- $\tilde{X}_t = \tilde{X}_0 + \int_0^t b(s, \tilde{X}_s, \rho_s(\tilde{X}_s))\mathrm{d}s + \sqrt{2}\tilde{W}_t.$

Uniqueness

- It is well-known that the following SDE is well-posed when B is bounded measurable (see [1])

$$\mathrm{d}X_t = B(t, X_t)\mathrm{d}t + \mathrm{d}W_t.$$

- Weak uniqueness of DDSDE(1.3)⇒Strong uniqueness of DDSDE(1.3).
- For any two solution X_t^1 and X_t^2 of DDSDE(1.3) with the same initial, we only need to prove they have the same density. Denote by $\rho_t^i(x)$ the density of X_t^i for $i = 1, 2$.
- $\rho_t^1(x)$ and $\rho_t^1(x)$ are also two solutions of NFPE (1.4). Noting that b is Lipschitz, we shall use Gronwall's inequality to get the uniqueness. However, we have to deal with $\int_0^T \|\rho_t^1\|^2_{L^\infty(\mathbb{R}^d)}\mathrm{d}t$. If we only use the Duhamel's formula in last page, it will blow up.
- By the heat kernel estimate, we have

$$\|\rho_t^1(\cdot)\|_{L^\infty} \leqslant Ct^{-d/2}\|\int_{\mathbb{R}^d} e^{-\frac{\cdot - y}{\lambda t}}\rho_0(y)\mathrm{d}y\|_{L^\infty}$$
$$\lesssim t^{-d/(2q)}\|\rho_0\|_{L^q}.$$

- Therefore, if $q > d$, we obtain the uniqueness.

[1] Veretennikov, A., On the strong solutions of stochastic differential equations. Theory Probab. Appl., 24 (1979), 354-366.

Part 2 : Rate of convergence of Euler approximation for SDEs driven by cylindrical α-stable processes.

Based on the joint work with Mingyan Wu.

α-stable process

- It is a well-known that every Lévy process L_t has a Lévy symbol Ψ, i.e.

$$\mathbb{E}e^{izL_t} = e^{t\Psi(z)}, \quad \forall z \in \mathbb{R}^d.$$

- Let $\alpha \in (0,2)$, a $\mathbb{R}^d$-valued Lévy process L_t is called a d-dimensional α-stable process if the Lévy symbol Ψ has the following representation:

$$\Psi(z) = \int_{\mathbb{R}^d} [e^{izx} - 1 - izx1_{|x|<1}]\nu(\mathrm{d}x),$$

where ν is called Lévy measure of L_t and

$$\nu(A) := \int_{\mathbb{S}^{d-1}} \mu(\mathrm{d}\omega) \int_0^\infty 1_A(r\omega) \frac{\mathrm{d}r}{r^{1+\alpha}}, \quad \forall A \in \mathcal{B}(\mathbb{R}^d),$$

$\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d; |x| = 1\}$ and μ is a finite measure on $(\mathbb{S}^{d-1}, \mathcal{B}(\mathbb{S}^{d-1}))$.

α-stable process

- In the sequel, all μ is non-degenerate, i.e.

$$\inf_{\tilde{\omega}\in\mathbb{S}^{d-1}}\int_{S^{d-1}}|\tilde{\omega}\cdot\omega|^2\mu(\mathrm{d}\omega)>0.$$

- For simplicity, we assume that μ is symmetric, i.e. $\mu(A)=\mu(-A)$.
- The infinitesimal generator $\mathscr{L}^\alpha$ of α-stable process L_t is

$$\mathscr{L}^\alpha f(x):=\text{p.v.}\int_{\mathbb{R}^d}(f(x+y)-f(x))\nu(\mathrm{d}y),$$

where p.v. is Cauchy principle value. It is a nonlinear operator.

Example 6

- When μ is the Lesbegue measure on $\mathbb{S}^{d-1}$, $\nu(\mathrm{d}y)=1/|y|^{d+\alpha}\mathrm{d}y$ and $\Psi(z)=-C|z|^\alpha$ with some absolute constant $C>0$.
- This time, we call L_t a standard d-dim α-stable process.
- Denote by $\Delta^{\alpha/2}$ the infinitesimal generator of L_t.

Cylindrical α-stable process

- Let $\{L_t^i\}_{i=1}^d$ be i.i.d. 1-dim standard α-stable processes.
- As you know, $(B_t^1, ..., B_t^d)$ is a d-dim BM when $\{B_t^i\}_{i=1}^d$ are i.i.d. 1-dim BMs. Is $(L_t^1, ..., L_t^d)$ a d-dim standard α-stable process?
- The answer is **NO!**

Example 7

- Let $L_t := (L_t^1, ..., L_t^d)$. Then $\mu = \sum_{i=1}^d \delta_{e_i}$ where δ is the Dirac measure and $e_i = (0, .., 1_{ith}, .., 0)$,

$$\nu(\mathrm{d}x) = \sum_{k=1}^{n} \delta_0(\mathrm{d}x_1) \cdots \delta_0(\mathrm{d}x_{k-1}) \frac{\mathrm{d}x_k}{|x_k|^{1+\alpha}} \delta_0(\mathrm{d}x_{k+1}) \cdots \delta_0(\mathrm{d}x_d).$$

- This time, we call L_t a **cylindrical** d-dim α-stable process.

- Notice that the Lévy measure of cylindrical α-stable process is even not absolute to Lesbegue measure.
- Cylindrical α-stable process is much more singular then the standard one.

SDEs driven by α-stable process

- Consider the following SDEs:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}L_t,$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and L_t is a α-stable process and following parabolic equation

$$\partial_t u = \mathscr{L}^\alpha_\sigma u + b \cdot \nabla u + f,$$

where

$$\mathscr{L}^\alpha_\sigma f(x) := \text{p.v.} \int_{\mathbb{R}^d} (f(x + \sigma(x)y) - f(x))\nu(\mathrm{d}y).$$

- Let $f \equiv 0$ and X^x_t be the solution SDE with $X^x_0 = x$. By Itô's formula, $u(\cdot - s, X^x_\cdot)$ is a martingale and

$$\mathbb{E}u_0(X^x_t) = \mathbb{E}u(t, X^x_0) = u(t, x).$$

- In the sequel, we assume that it is elliptical, i.e.

$$\inf_x \det \sigma(x) > 0.$$

SDEs driven by cylindrical α-stable process

- Also consider the Euler scheme:

$$\mathrm{d}X_t^N = b(X_{\phi_N(t)})\mathrm{d}t + \sigma(X_{\phi_N(t)})\mathrm{d}L_t.$$

- When σ and b are Lipschitz, it is easy to obtain the well-posed result and rate of convergence of Euler approximation for it. what if b is only in some Hölder space?
- It is well-known that ODE $X_t = \int_0^t b(X_s)\mathrm{d}s$ may be ill-posed when b is only Hölder continuous.
- To answer this question, I will introduce Schauder's estimate and Zvonkin's transform.

Schauder's estimate

- Let $a_{i,j}$ and b_k be measurable functions from $\mathbb{R}^d$ to $\mathbb{R}$, where $i, j, k \in \{1, 2, ..., d\}$. Define vector-valued function $b = (b_1, b_2, ..., b_d)$, and consider the following elliptic equation:

$$\sum_{i,j=1}^{d} a_{i,j}\partial_i\partial_j u + b \cdot \nabla u = f, \tag{2.1}$$

where $b \cdot \nabla u := \sum_{i=1}^{d} b_i \partial_i u$. Suppose that the source term $f \in \mathbf{C}^{\beta}(\mathbb{R}^d)$.

- Assume $a_{i,j}$ are elliptic,

$$\sum_{i,j=1}^{d} \xi_j a_{i,j} \xi_j \geqslant \lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^d,$$

and the relevant norms of coefficients are all bounded by another constant $\Lambda > 0$, i.e.,

$$\sum_{i,j=1}^{d} \|a_{i,j}\|_{\mathbf{C}^{\beta}(\mathbb{R}^d)} + \sum_{i=1}^{d} \|b\|_{\mathbf{C}^{\beta}(\mathbb{R}^d)} \leqslant \Lambda.$$

- Schauder's estimate: there is a positive constant $c = c(d, \beta, \lambda, \Lambda)$ such that for all solution $u \in \mathbf{C}^{2+\beta}(\mathbb{R}^d)$ of (2.1),

$$\|u\|_{\mathbf{C}^{2+\beta}(\mathbb{R}^d)} \leqslant c(\|u\|_{L^\infty(\mathbb{R}^d)} + \|f\|_{\mathbf{C}^{\beta}(\mathbb{R}^d)}).$$

Zvonkin's transform

- For simplification, we consider the following SDE:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}W_t, \tag{2.2}$$

  where $b : \mathbb{R}^d \to \mathbb{R}^d$ is Hölder and W_t is a standard BM.
- We consider the following backward PDE:

$$\partial_t u + \Delta u + b \cdot \nabla u + b = \lambda u, \quad u(T) = 0.$$

  By Schauder's estimate,

$$\|u\|_{\mathbb{L}^\infty_T \mathbf{C}^{2+\beta}} \leqslant C_T(\lambda)\|b\|_{\mathbf{C}^\beta} \quad C_T(\lambda) \to 0 \quad (\lambda \to \infty), \tag{2.3}$$

  where $\mathbb{L}^\infty_T \mathbf{C}^{2+\beta} := L^\infty([0, T]; \mathbf{C}^{2+\beta}(\mathbb{R}^d))$.
- Then, $\Phi_t(x) := u(t, x) + x$ is a diffeomorphism on $\mathbb{R}^d$ for some large λ and $Y_t := \Phi_t(X_t)$ satisfies the following SDE

$$\mathrm{d}Y_t = \nabla u(t, \Phi_t^{-1}(Y_t))\mathrm{d}W_t + \mathrm{d}W_t + \lambda u(t, \Phi_t^{-1}(Y_t))\mathrm{d}t.$$

Zvonkin's transform

- We consider the following Euler scheme:

$$\mathrm{d}X_t^N = b(X_{\phi_N(t)})\mathrm{d}t + \mathrm{d}W_t,$$

  and let $\lambda = 0$.
- By Itô's formula, we have

$$\begin{aligned}\Phi_t(X_t) - \Phi_t(X_t^N) = \Phi_s(X_s) - \Phi_s(X_s^N) \\ - \int_s^t [u(r, X_r) - u(r, X_r^N)]\mathrm{d}W_r \\ + \int_s^t [b(X^N_{\phi_N(r)}) - b(X_r^N)][\mathbb{I} - \nabla u(r, X_r^N)]\mathrm{d}r.\end{aligned}$$

- Noting that

$$\mathbb{E}|X_t^N - X^N_{\phi_N(t)}|^p \leqslant CN^{-p/2},$$

  and u, ∇u are Lipschitze, by some Gronwall-type inequality, we obtain the rate of

$$\mathbb{E}|X_t^N - X_t|^p.$$

- A natural question is whether Schauder's estimates hold when we replace the local operator $a_{ij}\partial_i\partial_j$ by some non-local ones?

Well-known results

- Actually, there are many known results.
- ★ For the following parabolic equation:

$$\partial_t u = \mathscr{L}_\sigma^\alpha u + b \cdot \nabla u + f, \quad u_0 = 0.$$

2012 (Silverstre, *Indi- ana Univ. Math. J., 61(2012)*)

- $\alpha \in (0,2)$, $\mathscr{L}_\sigma^\alpha = \Delta^{\alpha/2}$ and $b \in \mathbf{C}^\beta$ with $\alpha + \beta > 1$.

$$\|u\|_{\mathbb{L}_T^\infty \mathbf{C}^{\alpha+\beta}} \leqslant C_T \|f\|_{\mathbb{L}_T^\infty \mathbf{C}^\beta}.$$

2019 (Chaudru, Menozzi and Priola, *J. Funct. Anal. 128 (2020)*)

- $\alpha \in (1/2, 1)$, $\sigma \equiv \mathbb{I}$ and $b \in \mathbf{C}^\beta$ with $\alpha + \beta > 1$.
- ★ For the following elliptic equation:

$$\mathscr{L}_\sigma^\alpha u + b \cdot \nabla u = f.$$

2010 (Priola, *Osaka J. Math., 49 (2012)*)

- $\alpha \in (1,2)$, $-\Psi(z) \geqslant c|z|^\alpha$, $\sigma \equiv \mathbb{I}$ and $b \in \mathbf{C}^\beta$ with $\alpha + \beta > 1$.

2019 (Ling and Zhao, *arXiv:1907.00588*)

- $\alpha \in (0,1)$, $\nu(\mathrm{d}y) = 1/|y|^{d+\alpha}\mathrm{d}y$ and $\sigma, b \in \mathbf{C}^\beta$ with $\alpha + \beta > 1$.

2019 (Kühn, *Integral Equations Operator Theory 91(2019)*)

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$$
\begin{aligned}
\mathrm{d}X_t &= b(X_t)\mathrm{d}t + \sigma(X_{t-})\mathrm{d}L_t,\\
\mathrm{d}X_t^N &= b(X^N_{\phi_N(t)})\mathrm{d}t + \sigma(X^N_{\phi_N(t)})\mathrm{d}L_t.
\end{aligned}
$$

- Based on the Schauder's estimate for the non-local equation, there are some works about the Euler scheme.

2017 (Mikulevičius and Xu)

- Assume $\alpha \in [1,2)$, $\nu(\mathrm{d}y) = \rho(y)/|y|^{d+\alpha}$ with $c \leqslant \rho(y) \leqslant c^{-1}$, $\forall y \in \mathbb{R}^d$, $\rho(\lambda y) = \rho(y)$, σ is bounded Lipschitz and $b, \rho \in \mathbf{C}^\beta$ with $\beta > 1 - \alpha/2$. For any $p \in (0, \alpha)$, they have

$$
\mathbb{E}\left[\sup_{0\leqslant t\leqslant 1} |X_t^N - X_t|^p\right] \leqslant CN^{-p\beta/\alpha}.
$$

- Notice that they can not deal with the cylindrical case and $\alpha > 1$. Condition $\beta > 1 - \alpha/2$ is to guarantee the well-posed for the SDE.

2017 (Huang and Liao, *Stochastic Analysis and Applications, 36(2018)*)

- Assume $\alpha \in [1,2)$, $-\Psi(z) \geqslant c|z|^\alpha$ and $b, \rho \in \mathbf{C}^\beta$ with $\beta \in (1 - \alpha/2, 1)$. For any $p \in (0, \alpha/\beta)$, they have

$$
\mathbb{E}\left[\sup_{0\leqslant t\leqslant 1} |X_t^N - X_t|^p\right] \leqslant CN^{-p\beta/\alpha}.
$$

- Notice that they also can not deal with the case $\alpha < 1$.

Our results

- In fact, $\alpha < 1$ is the supercritical case. When $\alpha < 1$, the transport item, which has no regularity, dominates the diffusion items $\mathscr{L}_\sigma^\alpha$:

$$\partial_t u = \mathscr{L}_\sigma^\alpha u + b \cdot \nabla u + f, \quad u_0 = 0. \tag{2.4}$$

- There is no results about the Schauder's estimate for it when $b \in \mathbf{C}^\beta$, L_t is cylindrical and $\sigma \neq \mathbb{I}$.

Theorem 8 (Schauder's estimates)

Suppose that $\alpha \in (1/2, 1)$, μ is non-degenerate, σ is elliptical, $\sigma \in \mathbf{C}^\gamma$ with $\gamma \in (0, 1]$, $b \in \mathbf{C}^\beta$ with $\beta \in (1 - \alpha, \alpha\gamma)$, and $\alpha + \beta \notin \mathbb{N}$. For any $T > 0$, there is a constant $c > 0$ and a unique classical solution u of PDE (2.4) *satisfying,*

$$\|u\|_{\mathbb{L}_T^\infty(\mathbf{C}^{\alpha+\beta}(\mathbb{R}^d))} \leqslant c\|f\|_{\mathbb{L}_T^\infty(\mathbf{C}^\beta(\mathbb{R}^d))}.$$

- Condition $\alpha > 1/2$ comes from the condition $\beta \in (1 - \alpha, \alpha\gamma)$ which means

$$1 - \alpha < \alpha \Rightarrow \alpha > 1/2.$$

- We used a method based on Littlewood-Paley operators to prove it which can be find in [1] and [2].

[1] Hao, Z., Wu, M. and Zhang, X., Schauder estimates for nonlocal kinetic equations and applications. J. Math. Pures Appl. 140 (2020) 139-184.

[2] Hao, Z., Wang, Z. and Wu, M., Schauder's estimates for nonlocal equations with singular Lévy measures. Available at arXiv:2002.09887.

Our results

Corollary 9

Assume $\alpha \in (1/2, 1)$, μ is non-degenerate, σ is elliptical, σ is Lipschitz and $b \in \mathbf{C}^\beta$ with $\beta \in (1 - \alpha/2, 1)$. For any $p \in (0, \alpha)$ and $T > 0$, there is a constant C such that for all $N \in \mathbb{N}$

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t^N - X_t|^p\right] \leqslant CN^{-p\beta}.$$

Thanks for your attention!