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Strong convergence of propagation of chaos for McKean-Vlasov SDEs with singular interactions

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McKean-Vlasov SDEs

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$$\mathrm{d}X_t = b(t, X_t, \mu_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t, \tag{1}$$

where μ_t is the time marginal distribution of the solution.

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- By Itô's formula, μ_t solves the following nonlinear Fokker-Planck equation (FPE) in the distributional sense:

$$\partial_t \mu_t = \frac{1}{2}\partial_i \partial_j \left(\sigma_{ik}\sigma_{jk}(t)\mu_t\right) - \mathrm{div}(b(t, \cdot, \mu_t)\mu_t). \tag{2}$$

  - $b(\mu) = b * \mu$: (Vorticity form of the 2D Navier-Stokes equation), (Vlasov-Poisson-Fokker-Planck equation, kinetic), (surface quasi-geostrophic (SQG) equation, fractional), ...
  - $b(x, \mu) = b(\frac{\mu(\mathrm{d}x)}{\mathrm{d}x}(x))$, quasilinear PDE: (Burgers' equation), (Porous medium equation), ... (distribution density dependent case), (Nemytskii-type)

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- (McKean 1966): propose the DDSDE (1) to investigate a class of nonlinear PDEs like Vlasov equation (McKean-Vlasov SDEs).

Nemytskii-type

- Distribution density dependent SDEs (dDSDEs)

$$dX_t = b(t, X_t, \rho_t(X_t))dt + \sigma(t, X_t)dB_t, \tag{3}$$

where ρ_t is the density of the time marginal distribution of the solution w.r.t. the Lebesgue measure.

- Barbu-Röckner *SIAM, SPDE, AOP, JFA* 2018-2023: Superposition principle

Nonlinear FPEs (2) $\to$ McKean-Vlasov SDEs (3).

- Singular interaction kernels: For the Nemytskii's type, we can consider it as a convolution with the Dirac measure:

$$b(x, \mu) = b(\delta * \mu(x)) = b(\frac{\mu(dx)}{dx}(x)).$$

N-particle system

- Consider the following moderately interacting N-particle system:

$$\mathrm{d}X_t^{N,i} = b\left(t, X_t^{N,i} + \frac{1}{N}\sum_{j=1}^{N} \phi_{\varepsilon(N)}(X_t^{N,i} - X_t^{N,j})\right)\mathrm{d}t + \mathrm{d}B_t^i,$$

  - $\phi_{\varepsilon(N)}$: local interaction kernel; $\frac{1}{N}$: mean-field scaling (critical);
  - $\phi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon)$ with some cut-off probability density function φ and $\varepsilon(N) \to 0$ as $N \to \infty$;
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- Applications: In plasma physics particles can represent ions and electrons in the Vlasov-Poisson equation (Vlasov *SPU* 1968); in biosciences they characterize the collective behavior of individuals (Simon-Olivera *JDDE* 2018) and describe the growth of cancer (Flandoli-Leimbach-Olivera *JMAA* 2019)......

Propogation of chaos

- Based on the the famous Law of Large Numbers, we expect that

$$\frac{1}{N}\sum_{j=1}^{N}\delta_{X_t^{N,j}} \to \mu_t.$$

- (Kac 1956), see also (Sznitman 1991)

$$\frac{1}{N}\sum_{j=1}^{N}\delta_{X_t^{N,j}} \to \mu_t \iff \underbrace{\mathbb{P}\circ(X_t^{N,1},\dots,X_t^{N,k}) \to \mu_t^{\otimes k} \quad \forall k\in\mathbb{N}}_{\text{Kac's chaos}}.$$

- (Propogation of chaos)
  If Kac's chaos holds for $t=0$, then it holds for $t>0$.
  (McKean 1967, Osada 1987, Shkolnikov *SPA* 2012, Jabin-Wang *Invent.* 2018, Lacker *ECP* 2018,......)
- Moderately interacting particle systems: (Oelschläger *PTRF* 1985), (Jourdain-Méléard *AIHP* 1998), ... (smooth φ)

Our aim

- Well-posedness (existence of the strong solution and pathwise uniqueness) for the following distribution density and distributional dependent SDE (dDDSDE):

$$dX_t = b(t, X_t, \rho_t(X_t), \mu_t)dt + \sigma(t, X_t)dB_t, \tag{4}$$

  and its N-particle system when b is singular (L^p).
- Propagation of chaos (qualitative and quantitative) for the moderately interacting particle systems.

Assumptions

- We define

$$\mathscr{I} := \{(p,q) \in (2,\infty) \mid \frac{d}{p} + \frac{2}{q} < 1\}$$

and assume for some $(p,q) \in \mathscr{I}$

$$c_0^{-1}|\xi| \le |\sigma(t,x)\xi| \le c_0|\xi|, \quad \|\nabla_x \sigma\|_{L_q^p(T)} < \infty, \tag{$\mathbf{H}_\sigma$}$$

where $L_q^p(T) := L^q([0,T]; L^p(\mathbb{R}^d))$.

Main results-(Well-posedness)

① **Theorem 1.**

If $(r,\mu)\to b(\cdot,r,\mu)$ is a Lipschitz function from $\mathbb{R}_+\times(\mathcal{P}(\mathbb{R}^d),\|\cdot\|_{var})\to L_q^p(T)$ with some $(p,q)\in\mathscr{I}$, then there is a unique strong solution to dDDSDE (4) .

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- Well-known results.
  - (Li-Min *SIAM* 2016, Wang *SPA* 2018): $(\mathcal{P}(\mathbb{R}^d),W_p)$;
  - (Mishura-Veretenikov *TPMS* 2020): bounded drift;
  - (Röckner-Zhang *Bernoulli* 2021, Zhao 2020): $L_q^p(T)$;
  - (Zhang *CMS* 2023): super-critical;
  - (Lacker *ECP* 2018, Han 2022): Entropy method;
  - (Wang *JDE* 2023): distribution density dependent case (Nemytskii's type).
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  - $\cdots$
- Idea of proof: Picard iteration

$$\mathrm{d}X_t^n = b(t, X_t^n, \rho_t^{n-1}(X_t^n), \mu_t^{n-1})\mathrm{d}t + \sigma(t, X_t^n)\mathrm{d}B_t.$$

  - Convergence of μ^n: Entropy method;
  - Convergence of ρ^n: Uniform estimates for FPE.

Main results

- We consider the following N-particle system:

$$\mathrm{d}X_t^{N,i} = b\left(t, X_t^{N,i}, \frac{1}{N}\sum_{j=1}^{N}\phi(t, X_t^{N,i} - X_t^{N,j})\right)\mathrm{d}t + \sigma(t, X_t^{N,i})\mathrm{d}B_t^i. \tag{5}$$

  - For $b(t,x,y) = y$, the related limit DDSDE is

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  - For $\phi(t,x) = \varepsilon^{-d}\varphi(x/\varepsilon)$ with some $\varepsilon = \varepsilon(N) \to 0$ as $N \to \infty$, the related limit DDSDE is the Nemytskii-type:

$$\mathrm{d}X_t = b(t, X_t, \rho_t(X_t))\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t.$$

Main results-(Well-posedness)

② **Theorem 2.**

Assume for some function $h \in L_q^p(T)$ with some $(p,q) \in \mathscr{I}$ and constant $\kappa_1 > 0$

$$|b(t,x,y)| \leq h(t,x) + \kappa_1 |y|, \quad |b(t,x,y_1) - b(t,x,y_2)| \leq \kappa_1 |y_1 - y_2|. \tag{H$_b$}$$

Given any $\phi \in L_q^p(T)$, there is a unique strong solution to (5).

▷ This is an extension result to those in (Krylov-Röckner *PTRF* 2005), since for $B_i(t,x_1,..,x_N) := \frac{1}{N}\sum_{j=1}^N \phi(t, x_i - x_j)$ **cannot** satisfy

$$\frac{Nd}{p} + \frac{2}{q} < 1, \quad N \to \infty.$$

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- Idea of the proof: Introduce the mixed L^p space with permutation.
  We note that for different $i = 1,..,N$, B_i are in different mixed L^p spaces. Thus the result in (Ling-Xie *POTA* 2022) cannot be valid here.
- Difficulty and innovation: Obtain the maximal regularity of the solution to the related PDE in a sum space and construct the Zvonkin transformation under this mixed L^p space with permutation setting.

Main results-(Propagation of chaos)

▷ Consider the N-independent copies of the limit McKean-Vlasov SDE:

$$\mathrm{d}X_t^i = b(t, X_t, \phi * \mu_t(t, X_t^i))\mathrm{d}t + \sigma(t, X_t^i)\mathrm{d}B_t^i.$$

③ **Theorem 3.**

Assume (H_b) holds. Given any $\phi \in L_q^p(T)$ with some $(p, q) \in \mathscr{I}$, propagation of chaos holds. Moreover, if ϕ is bounded and $(X_0^{N,1}, \ldots, X_0^{N,N}) = (X_0^1, \ldots, X_0^N)$, we have the following optimal rate for the path estimate:

$$\sup_i \mathbb{E}\left(\sup_{t\in[0,T]} |X_t^{N,i} - X_t^i|\right) \leq \frac{Ce^{CT\|\phi\|_\infty}}{\sqrt{N}}.$$

- ▷ This gives a path version of propagation of chaos for singular interaction kernels.
- ▷ The nonlinearity of $\mu \to b(t, x, \phi * \mu)$ makes some technical trouble.
- ▷ The independence of the initial data is not necessary for the propagation of chaos when $\phi \in L_q^p(T)$ (strong solution).

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③ **Theorem 3.**

Assume (H_b) holds and $(X_0^{N,1}, ..., X_0^{N,N}) = (X_0^1, ..., X_0^N)$. For any bounded probability density function φ and $\phi(t,x) = \varepsilon^{-d}\varphi(x/\varepsilon)$ there is some $\beta > 0$

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_t^{N,i} - X_t^i|\right) \leqslant C\mathrm{e}^{C\varepsilon_N^{-2d}}\frac{1}{N} + C\varepsilon_N^{\beta}.$$

- ▷ When $\varepsilon \sim (\ln N)^{-\delta}$ with some $\delta > 0$, we have the path convergence $X^{N,i} \to X^i$.
- ▷ Compared with the results in previous work for moderately interacting particle systems, we don't need the smooth assumptions on φ.
  We believe that this is useful for numerical experiments by taking $\varphi(x) := 1_{[0,1]}(x)$.

Future works

- (Ongoing work with Jean-François Jabir, Stéphane Menozzi, Michael Röckner and Xicheng Zhang)
  - Convergence rate of propagation of chaos for $W^{-\beta,p}$ kernels and second order system:

$$\begin{cases} \mathrm{d}X_t^{N,i} = V_t^{N,i}\mathrm{d}t, \\ \mathrm{d}V_t^{N,i} = \frac{1}{N}\sum_{j\neq i}(K*\phi_N)(t, X_t^{N,i} - X_t^{N,j})\mathrm{d}t + \sqrt{2}\mathrm{d}B_t^i, \end{cases}$$

    where $i = 1, 2, .., N$,
    - $(X^{N,i}, V^{N,i}) \in \mathbb{R}^{2d}$: position and velocity of particle number i;
    - K: Poisson kernel, Riestz kernel, ...
    - $\phi_N(t,x,v) = N^{4\vartheta d}\varphi(N^{3\vartheta}(x+tv), N^{\vartheta}v)$ with some $\vartheta < 1/(4d)$;
  - Propagation of chaos for McKean-Vlasov SDEs driven by α-stable processes.

Thank you !