Gradient estimate for SDEs driven by cylindrical Lévy processes

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Outline

- Introduction
- Main results
- Proof
- Future works

Part 1 : Introduction

Motivation

- Let $d \geqslant 2$. Consider the following stochastic differential equation :

$$\begin{cases} \mathrm{d}X_t = A(X_t)\mathrm{d}B_t + b(X_t)\mathrm{d}t, \\ X_0 = x \in \mathbb{R}^d \end{cases} \tag{1.1}$$

where $B_t = (B_t^1, ..., B_t^d)$ is a d-dimensional standard Brownian motion, $b : \mathbb{R}^d \to \mathbb{R}^d$ is a measurable function, and $A : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a $d \times d$ matrix-valued measurable function and satisfies

(H) $A \in C(\mathbb{R}^d)$ and for some $c_0 \geqslant 0$, it holds that

$$|\det A(x)| \geqslant c_0, \quad x \in \mathbb{R}^d.$$

- Under the above assumptions and b is bounded, it is well known that for each $x \in \mathbb{R}^d$, SDE (1.1) admits a unique weak solution $X_t(x)$(see [1]) . Furthermore, if A and b have more regularity it admits a density $p_t(x, y)$ enjoying the following estimates(see [2]): for any $T > 0$, there are constants $c_i > 0$ such that for all $0 < t < T$ and $x, y \in \mathbb{R}^d$

$$c_1 t^{-d/2} e^{-c_2|x-y|^2/t} \leqslant p_t(x, y) \leqslant c_3 t^{-d/2} e^{-c_4|x-y|^2/t}.$$

[1] Bass, R.F., Diffusions and Elliptic Operators. Springer-Verlag, New York, 1997
[2] Z.-Q. Chen, E. Hu, L. Xie, and X. Zhang, Heat kernels for non-symmetric diffusion operators with jumps.*J. Differential Equations, 263 (2017), 6576-6634.*

- Notice that B_t^i in $B_t = (B_t^1, ..., B_t^d)$ are i.i.d. 1-dimensional standard Brownian motions.
- Naturally, we consider the standard cylindrical α-stable process $L_t = (L_t^1, ..., L_t^d)$ and the following SDE

$$\begin{cases} \mathrm{d}X_t = A(X_{t-})\mathrm{d}L_t + b(X_t)\mathrm{d}t, \\ X_0 = x \in \mathbb{R}^d, \end{cases} \tag{1.2}$$

  where L_t^i are i.i.d. 1-dimensional standard α-stable processes.
- In fact, L_t admits a density $p_t(x)$ enjoying the following estimates :
  For any $T > 0$, there are constants $c_1, c_2 > 0$ such that for all $0 < s < t < T$ and $x \in \mathbb{R}^d$

$$c_1 \Pi_{i=1}^d \frac{t}{(\sqrt{t} + |x_i|)^{\alpha+1}} \leqslant p_t(x) \leqslant c_2 \Pi_{i=1}^d \frac{t}{(\sqrt{t} + |x_i|)^{\alpha+1}}.$$

- However, there is no result for the density estimate for X_t. Actually, the existence of the solution X_t and the density of X_t are still problem.

- More generality, we consider the following SDE driven by the cylindrical α-stable process L_t,

$$\begin{cases} \mathrm{d}X_t^x = \int_{\mathbb{R}^d} \sigma(X_{t-}, z) N(\mathrm{d}t, \mathrm{d}z) + b(X_t)\mathrm{d}t, \\ X_0^x = x \in \mathbb{R}^d, \end{cases} \tag{1.3}$$

where $\sigma = (\sigma_i)_{i=1}^d : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a measurable function, and $N(\mathrm{d}t, \mathrm{d}z)$ is the Poisson random measure of L_t^α defined as follow

$$N((s,t], E) := \sum_{s<u\leqslant t} \mathbf{1}_{(L_u - L_{u-}) \in E}.$$

- Define $\nu(E) := \mathbb{E}N([0,1], E)$. For simplify, we assume that for all $x \in \mathbb{R}^d$ and $0 < r < R < +\infty$

$$\int_{r\leqslant|z|\leqslant R} \sigma(x,z)\nu(\mathrm{d}z) = 0.$$

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Questions:

- In what condition of σ and b, there is a weak(or strong) solution of SDE (1.3)?
- If there is a weak solution, does the solution have a density?
- If there is a weak solution, can we get some precise estimates for it?

- When L_t is a d-dimensional standard α-stable process, the infinitesimal generator of X_t^x has the following form

$$\begin{aligned}\mathscr{L} f(x) &= \text{p.v.} \int_{\mathbb{R}^d} \frac{f(x+\sigma(x,z)) - f(x)}{|z|^{d+\alpha}} \mathrm{d}z \\ &= \text{p.v.} \int_{\mathbb{R}^d} \frac{f(x+z) - f(x)}{|z|^{d+\alpha}} \kappa(x,z) \mathrm{d}z,\end{aligned} \tag{1.4}$$

  where

$$\kappa(x,z) = \frac{|z|^{d+\alpha}}{|\sigma^{-1}(x,z)|^{d+\alpha}} |\det \nabla_z \sigma^{-1}(x,z)|.$$

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  where

$$\kappa(x,z) = \frac{|z|^{d+\alpha}}{|\sigma^{-1}(x,z)|^{d+\alpha}}|\det \nabla_z \sigma^{-1}(x,z)|.$$

- When L_t is a d-dimensional cylindrical α-stable process, which is our case, the infinitesimal generator of X_t^x has the following form

$$\mathscr{L}f(x) = \sum_{i=1}^{d} \text{p.v.}\int_{\mathbb{R}} \frac{f(x+\sigma(x,ze_i)) - f(x)}{|z|^{1+\alpha}}\mathrm{d}z,$$

  where $e_i = (0, .., 1(\text{i-}th), .., 0)$.

- Notice that, it is impossible to find such a κ in (1.4) this time.

- Let $\mathscr{F}$ be the Fourier transform. The infinitesimal generator of d-dimensional cylindrical α-stable process is $\sum_{i=1}^{d}(\partial_i\partial_i)^{\frac{\alpha}{2}}$ with

$$\mathscr{F}(\sum_{i=1}^{d}(\partial_i\partial_i)^{\frac{\alpha}{2}}f)(\xi) = c\sum_{i=1}^{d}|\xi_i|^{\alpha}\mathscr{F}(f)(\xi) := \psi_1(\xi)\mathscr{F}(f)(\xi),$$

where $\psi_1 \in C^{\infty}(\mathbb{R}^d \setminus (\cup_{i=1}^{d}\mathbb{R}_i))$, where

$$\mathbb{R}_i := \{x \in \mathbb{R}^d; x_i = 0\}.$$

- The infinitesimal generator of d-dimensional standard α-stable process is $\Delta^{\frac{\alpha}{2}}$ with

$$\mathscr{F}(\Delta^{\frac{\alpha}{2}}f)(\xi) = c|\xi|^{\alpha}\mathscr{F}(f)(\xi) := \psi_2(\xi)\mathscr{F}(f)(\xi),$$

where $\psi_2 \in C^{\infty}(\mathbb{R}^d \setminus 0)$.

- Therefore, compared with standard α-stable process, the cylindrical one is more difficult to be dealed with.

Assumptions

(A$^\sigma$) $\sigma(x,z) = A(x)z$ for some matrix value map $A = (a_{i,j}) : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, there is a positive number c_0 such that for any $x, y, \xi \in \mathbb{R}^d$ and all $i, j = 1, ..., d$

$$c_0^{-1}|\xi| \leqslant |\xi \cdot A(x)\xi| \leqslant c_0|\xi|, \tag{1.5}$$

$$|a_{i,j}(x) - a_{i,j}(y)| \leqslant c_0|x - y|. \tag{1.6}$$

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(A$^b_\beta$) For $\beta \in (0,1)$,

$$\|b\|_{C^\beta} := \sup_{x\in\mathbb{R}^d} |b(x)| + \sup_{|x-y|\neq 0} \frac{|b(x)-b(y)|}{|x-y|^\beta} < \infty. \tag{1.7}$$

- We always assume that there is a weak solution X_t^x of SDE (1.3) and define

$$P_t^{\sigma,b}\phi(x) = \mathbb{E}(\phi(X_t^x)), \qquad P_t^\sigma := P_t^{\sigma,0}.$$

Well-known results

2006 (Bass-Chen)

There is a weak solution X_t^x of (1.3) when $\sigma(x,z) = \sigma(x)z$ is continuous in variable x and $b \equiv 0$.

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Assume $\sigma(x,z) = \sigma(x)z$ is continuous in variable x and $b \equiv 0$. For any bounded domain $D \subset \mathbb{R}^d$, define $\tau_D := \inf\{t > 0,\ X_t^x \notin D\}$. If any bounded function h satisfies

$$h(x) = \mathbb{E}[h(X_{\tau_D}^x)] \quad \text{for every } x \in D,$$

then h is Hölder continuous in D.

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2012 (Debussche-Fournier)

2017 (Chen-Zhang-Zhao)
Under the condition **($\mathbf{A}^\sigma$)** and **($\mathbf{A}_\beta^b$)** with $\beta \in (1 - \frac{\alpha}{2}, 1)$, there is a unique strong solution of (1.3).

2018 (Kulczycki-Ryznar-Sztonyk)
Assume $b \equiv 0$ and L_t^ν is a cylindrical α-stable process with $\alpha \in (0,1)$. Under the condition **(A$^\sigma$)**, for any $\gamma \in (0,\alpha)$, $T > 0$, there is a constant C such that for all $t \in (0,T]$, $x, y \in \mathbb{R}^d$ and $f \in L^\infty(\mathbb{R}^d)$

$$|P_t^\sigma f(x) - P_t^\sigma f(y)| \leqslant C|x-y|^\gamma t^{-\frac{\gamma}{\alpha}} \|f\|_{L^\infty}. \tag{1.8}$$

For any $\gamma \in (0, \frac{\alpha}{d})$, $T > 0$, there is a constant C such that for all $t \in (0,T]$, $x \in \mathbb{R}^d$ and $f \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$

$$|P_t^\sigma f(x)| \leqslant C t^{-\frac{\gamma d}{\alpha}} \|f\|_{L^\infty}^{1-\gamma} \|f\|_{L^1}^\gamma. \tag{1.9}$$

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- Notice that they can **not** deal the case $\alpha \in [1,2)$.
- Hölder index γ can **not** be 1.

Part 2: Our main results

Littlewood-Paley decomposition and Besov space

- Let ϕ_0 be a radial C^∞-function on $\mathbb{R}^d$ with
$$\phi_0(\xi) = 1 \text{ for } \xi \in B_1 \text{ and } \phi_0(\xi) = 0 \text{ for } \xi \notin B_2.$$
- For $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^d$ and $j \in \mathbb{N}$, define
$$\phi_j(\xi) := \phi_0(2^{-j}\xi) - \phi_0(2^{-(j-1)}\xi).$$
- It is easy to see that for $j \in \mathbb{N}$, $\phi_j(\xi) = \phi_1(2^{-(j-1)}\xi) \geqslant 0$ and
$$\mathrm{supp}\phi_j \subset B_{2^{j+1}} \setminus B_{2^{j-1}}, \quad \sum_{j=0}^{k} \phi_j(\xi) = \phi_0(2^{-k}\xi) \to 1, \quad k \to \infty.$$
- Notice that $\{\phi_j\}_{j\in\mathbb{N}_0}$ is a partition of unity of
$$\mathbb{R}^d = B_2 \cup \Big(\cup_{j\in\mathbb{N}} \left(B_{2^{j+1}} \setminus B_{2^{j-1}}\right)\Big).$$

- For given $j \in \mathbb{N}_0$, the block operator Δ_j is defined on $\mathscr{S}'$ by

$$\Delta_j f(x) := \mathscr{F}^{-1}(\phi_j \mathscr{F}(f))(x) = \mathscr{F}^{-1}(\phi_j) * f(x)$$
$$= 2^{\cdot m(j-1)} \int_{\mathbb{R}^d} \mathscr{F}^{-1}(\phi_1)(2^{(j-1)}(x-y)) f(y) \mathrm{d}y.$$

- For $j \in \mathbb{N}_0$, by definition it is easy to see that

$$\Delta_j = \Delta_j \widetilde{\Delta}_j, \text{ where } \widetilde{\Delta}_j := \Delta_{j-1} + \Delta_j + \Delta_{j+1} \text{ with } \Delta_{-1} \equiv 0, \qquad (2.1)$$

and Δ_j is symmetric in the sense that

$$\langle \Delta_j f, g \rangle = \langle f, \Delta_j g \rangle.$$

- The cut-off low frequency operator S_k is defined by

$$S_k f := \sum_{j=0}^{k-1} \Delta_j f = 2^{dk} \int_{\mathbb{R}^d} \check{\phi}_0(2^k(x-y)) f(y) \mathrm{d}y \to f. \qquad (2.2)$$

- We rewrite (2.2) as

$$f = \sum_{j=0}^{\infty} \Delta_j f,$$

which is called the Littlewood-Paley decomposition.

Definition 1 (Besov space)

For any $s \in \mathbb{R}$ and $p \in [1, \infty]$, the Besov space $B^s_{p,\infty}$ is defined by

$$B^s_{p,\infty}(\mathbb{R}^d) := \left\{ f \in \mathscr{S}'(\mathbb{R}^d) : \|f\|_{B^s_{p,\infty}} := \sup_{j \geqslant 0} \left(2^{sj} \|\Delta_j f\|_{L^p} \right) < \infty \right\}.$$

Proposition 2

For any $s_1 \geqslant 0$ *and* $s_2 > 0$ *with* $s_2 \notin \mathbb{N}$*,*

$$H^{s_1,p}(\mathbb{R}^d) \subset B^{s_1}_{p,\infty}(\mathbb{R}^d) \quad \textit{and} \quad C^{s_2}(\mathbb{R}^d) = B^{s_2}_{\infty,\infty}(\mathbb{R}^d),$$

where $H^{s_1,p}(\mathbb{R}^d)$ *and* $C^{s_2}(\mathbb{R}^d)$ *are the common Sobolev space and Hölder space respectively.*
For any $n \in \mathbb{N}$*,*

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For any $n \in \mathbb{N}$,

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Our assumption for σ

($\mathbf{H}_\mu^\sigma$) There is a constant $c_0 > 1$ such that for all $x, y, z \in \mathbb{R}^d$ and all $\lambda > 0$

$$\inf_{\omega\in\mathbb{S}^{d-1}} \inf_{\lambda>0} \lambda \sum_{i=1}^{d} |\omega \cdot \sigma(x, \frac{e_i}{\lambda})| \geqslant c_0^{-1}, \tag{2.3}$$

$$|\sigma(x, z) - \sigma(y, z)| \leqslant c_0|x - y||z|.$$

$$c_0^{-1}|z| \leqslant |\sigma(x, z)| \leqslant c_0|z|.$$

Remark 3

- *Notice that condition $\boldsymbol{H}_s^\sigma$ implies condition $\boldsymbol{H}_\mu^\sigma$ here.*
- $\sigma(x, z) = (2 + sin z_1)z$ *satisfies condition $\boldsymbol{H}_\mu^\sigma$ but not satisfies condition $\boldsymbol{H}_s^\sigma$.*

Main Results

Theorem 4

Let $\alpha \in (0,2)$ *and* $\beta \in [0,1]$ *with* $\alpha+\beta>1$. *Assume* $(\mathbf{H}^\sigma)$, $\|\nabla\sigma\|_\infty \leqslant c_1$ *for some* $c_1>0$, *and one of the following conditions holds:*

$$\text{(i) } b=0,\ \beta=1;\ \ \text{(ii) } \alpha\in(\tfrac{1}{2},2) \text{ and } b\in L^\infty(\mathbb{R}_+;\mathbf{C}^\beta).$$

Let $X_{s,t}(x)$ *be the unique solution of SDE* (**??**) *and define*

$$P_{s,t}\varphi(x) := \mathbb{E}\varphi(X_{s,t}(x)).$$

Let $\gamma\in[0,\alpha+\alpha\wedge\beta)$ *and* $\eta\in(-((\alpha+\beta-1)\wedge 1),\gamma]$. *For any* $T>0$, *there exists a constant* $C>0$ *such that for all* $0\leqslant s<t\leqslant T$,

$$\|P_{s,t}\varphi\|_{\mathbf{B}^\gamma_{\infty,\infty}} \leqslant C(t-s)^{\frac{\eta-\gamma}{\alpha}}\|\varphi\|_{\mathbf{B}^\eta_{\infty,\infty}}. \tag{2.4}$$

- Notice that (2.4) reduced the restriction of the γ in (1.8) from $(0,\alpha)$ to $(0,\alpha+\alpha\wedge\beta)$ by taking $\eta=0$. In particular, we have gradient estimate. Moreover, we can deal with the case $\alpha\geqslant 1$.
- By a way of interpolation, we also get (1.9).

Main Results

Corollary 5

(A) *Let* $\varphi \in \cup_{\eta<(\alpha+\beta-1)\wedge 1}\mathbf{B}^{-\eta}_{\infty,\infty}$. *For any* $0 \leqslant s < t$, $P_{s,t}\varphi \in \cap_{\gamma<\alpha+\alpha\wedge\beta}\mathbf{B}^{\gamma}_{\infty,\infty}$ *solves the following backward Kolmogorov equation: for all* $x \in \mathbb{R}^d$,

$$P_{t_0,t}\varphi(x) = P_{t_1,t}\varphi(x) + \int_{t_0}^{t_1} \mathscr{L}_s^{\sigma,b} P_{s,t}\varphi(x)\mathrm{d}s, \ \ 0 \leqslant t_0 < t_1 < t. \tag{2.5}$$

(B) *For* $\alpha \in (\frac{1}{2}, 2)$, *the following gradient estimate holds: for* $0 \leqslant s < t \leqslant T$,

$$\|\nabla P_{s,t}\varphi\|_\infty \leqslant C(t-s)^{-\frac{1}{\alpha}}\|\varphi\|_\infty. \tag{2.6}$$

(C) *For each* $s < t$, *the random variable* $X_{s,t}(x)$ *admits a density* $p_{s,t}(x,\cdot)$ *with*

$$p_{s,t}(x,\cdot) \in \cap_{\eta<(\alpha+\beta-1)\wedge 1}\mathbf{B}^{\eta}_{1,1}. \tag{2.7}$$

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$$P_{t_0,t}\varphi(x) = P_{t_1,t}\varphi(x) + \int_{t_0}^{t_1} \mathscr{L}^{\sigma,b}_s P_{s,t}\varphi(x)\mathrm{d}s, \ \ 0 \leqslant t_0 < t_1 < t. \quad (2.5)$$

(B) *For* $\alpha \in (\frac{1}{2}, 2)$, *the following gradient estimate holds: for* $0 \leqslant s < t \leqslant T$,

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- Notice that (**??**) reduced the restriction of the γ in (1.8) from $(0,\alpha)$ to $(0,\alpha+\alpha\wedge\beta)$. In particular, we have gradient estimate. Moreover, we can deal with the case $\alpha \geqslant 1$.
- By a way of interpolation, we also get (1.9) from Theorem**??**.

Part 3: Proof

PDE related to SDE

- Naturally we consider the following PDE,

$$\begin{cases}\partial_t u(t,x) = \mathscr{L}_\sigma^\alpha u(t,x) + b(x)\cdot\nabla u(t,x), \\ u(0,x) = \phi(x),\end{cases} \tag{3.1}$$

where $\phi \in C^\infty(\mathbb{R}^d)$ and

$$\mathscr{L}_\sigma^\alpha u(t,x) = \sum_{i=1}^d \text{p.v.} \int_{\mathbb{R}} \Big(u(t, x+\sigma(x,z)) - u(t,x)\Big)\nu(\mathrm{d}z).$$

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Definition 6

We call a function $u(t,x) \in L^\infty_{loc}([0,+\infty); C^{\alpha+\varepsilon}(\mathbb{R}^d) \cap C^{1+\varepsilon}(\mathbb{R}^d))$ for some $\varepsilon > 0$ be a classical solution of PDE (3.1) in $[0,T]$ if $u, \nabla u \in C_{loc}([0,\infty) \times \mathbb{R}^d)$ and for all $t \in [0,\infty)$ and $x \in \mathbb{R}^d$

$$u(t,x) = \int_0^t \mathscr{L}_\sigma^\alpha u(s,x) + b(x)\cdot\nabla u(s,x)\mathrm{d}s + \phi(x)$$

- Is there a classical solution of PDE (3.1)?
  - Fortunately, we have a priori estimate: under the condition $(\mathbf{H}^\sigma_\mu)$ and $(\mathbf{H}^b_\beta)$ with $\beta \in ((1-\alpha)\vee 0, \alpha)$, for any $T > 0$ and $\varepsilon \in (0, \beta \wedge \alpha)$, there is a constant C such that for all $t \in [0,T]$, $\phi \in C^\infty$ and classical solutions u

$$\|u(t)\|_{C^{\alpha+\varepsilon}} \leqslant C\|\phi\|_{C^{\alpha+\varepsilon}}. \tag{3.2}$$

  - By (3.2) some continuity methods and vanishing viscosity approach, we obtain the existence of the classical solution.

- Is there a classical solution of PDE (3.1)?
  - Fortunately, we have a priori estimate: under the condition $(\mathbf{H}^\sigma_\mu)$ and $(\mathbf{H}^b_\beta)$ with $\beta \in ((1-\alpha)\vee 0, \alpha)$, for any $T>0$ and $\varepsilon \in (0, \beta\wedge\alpha)$, there is a constant C such that for all $t\in[0,T]$, $\phi\in C^\infty$ and classical solutions u
$$\|u(t)\|_{C^{\alpha+\varepsilon}} \leqslant C\|\phi\|_{C^{\alpha+\varepsilon}}. \tag{3.2}$$
  - By (3.2) some continuity methods and vanishing viscosity approach, we obtain the existence of the classical solution.
- Let u be a classical solution. By Itô formula, $s \to u(t-s, X^x_s)$ is a martingale for $s\in[0,t]$. Then
$$P^{\sigma,b}_t\phi(x) = \mathbb{E}(\phi(X^x_t)) = \mathbb{E}(u(t-s, X^x_s)) = \mathbb{E}(u(t,x)) = u(t,x).$$
- The equality above tell us that if we want to establish any estimate of $P^{\sigma,b}_t\phi(x)$, it is enough to establish the estimate of classical solution u.
- Moreover, it tell us that the uniqueness of weak solution of SDE (1.3) is equivalent to the uniqueness of classical solution of PDE (3.1).

Crucial lemma

- Let $\theta : \mathbb{R}_+ \to \mathbb{R}^d$ is a measurable function and $p_{s,t}$ be the transition probability of process

$$Z_{s,t} := \int_s^t \int_{\mathbb{R}^d} \sigma(\theta(r), z)\tilde{N}(dz, dr).$$

Lemma 7 (Crucial Lemma)

- *For any* $\beta \in [0, \alpha)$, $\gamma \in [0, +\infty)$ *and* $T > 0$, *there is a constants* C *such that for* $m \in \mathbb{N}_0$ *all* $j > 0$, $f \in L^1_{loc}(\mathbb{R}_+)$ *and* $t \in (0, T]$ $s \in [0, t)$,

$$\int_0^t \int_{\mathbb{R}^d} |x|^\beta |\nabla^m \Delta_j p_{s,t}(x)||f(s)|\mathrm{d}x\mathrm{d}s \leqslant C2^{(m-\gamma-\beta)j} \int_0^t (t-s)^{-\frac{\gamma}{\alpha}} |f(s)|\mathrm{d}s.$$

- *For any* $m \in \mathbb{N}_0$, $q \in [1, \infty]$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\gamma \in [0, +\infty)$, *there is a constant* C *such that for all* $(t - s) \in (0, T]$,

$$\|\nabla^m \Delta_j p_{s,t}\|_{L^q(\mathbb{R}^d)} \leqslant C(t-s)^{-\frac{1}{\alpha}(\gamma - m + \frac{d}{p})} 2^{-\gamma j}.$$

The key point of proof

- For simplify, we assume $\sigma(x,z) = A(x)z$ for some matrix value map $A : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $a(t) := A(\theta(t))$. Recall that $p_{s,t}$ is the transition probability of

$$Z_{s,t} = \int_s^t a(r)\mathrm{d}L_t^\alpha \quad \text{with} \quad \lambda^{\frac{1}{\alpha}} L_{\lambda t}^\alpha \overset{(d)}{=} L_t^\alpha.$$

Therefore using the change of variable and the scaling property, we have

$$\int_s^t a(r)\mathrm{d}L_r^\alpha = \int_0^{t-s} a(r+s)\mathrm{d}\Big(L_{r+s}^\alpha - L_s^\alpha\Big)$$
$$\overset{(d)}{=} (t-s)^{-\frac{1}{\alpha}} \int_0^1 a(r(t-s)+s)\mathrm{d}L_r^\alpha.$$

We denote by $\bar{p}_{0,1}$ the density of $\int_0^1 a(r(t-s)+s)\mathrm{d}L_r^\alpha$, then

$$p_{s,t}(x) = (t-s)^{-\frac{d}{\alpha}} \bar{p}_{0,1}((t-s)^{-\frac{1}{\alpha}} x).$$

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- For simplify, we assume $\sigma(x,z) = A(x)z$ for some matrix value map $A : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $a(t) := A(\theta(t))$. Recall that $p_{s,t}$ is the transition probability of

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$$p_{s,t}(x) = (t-s)^{-\frac{d}{\alpha}} \bar{p}_{0,1}((t-s)^{-\frac{1}{\alpha}} x).$$

- Condition

$$\inf_{\omega \in \mathbb{S}^{d-1}} \inf_{\lambda > 0} \lambda \sum_{i=1}^d |\omega \cdot \sigma(x, \frac{e_i}{\lambda})| \geqslant c_0^{-1}, \tag{3.3}$$

guarantee that for any $n \in \mathbb{N}_0$ and $\beta \in [0,\alpha)$, there is a constant C such that

$$\int_{\mathbb{R}^d} |x|^\beta |\nabla^n \bar{p}_{0,1}(x)|\mathrm{d}x \leqslant C.$$

Our approach

- Firstly, we use a technology of translate alone the characteristic line θ_t and get a new equation:

$$\begin{cases} \partial_t \tilde{u}(t,x) = \mathscr{L}^{\alpha}_{\tilde{\sigma}} \tilde{u}(t,x) + \tilde{b}(x) \cdot \nabla \tilde{u}(t,x), \\ u(0,x) = \phi(x), \end{cases} \tag{3.4}$$

where $\tilde{u}(t,x) = u(t, x+\theta_t)$ and $\tilde{b}(x) = b(x+\theta_t) - b(\theta_t)$. Notice that $|\tilde{b}(x)| \lesssim |x|^\beta$ which releases the regularity of spatial x.

- Then we have the following presentation

$$\tilde{u}(t,x) = \int_0^t P_{s,t}\Big(\mathscr{L}^{\alpha}_{\tilde{\sigma}} - \mathscr{L}^{\alpha}_{\tilde{\sigma}_0}\Big)\tilde{u}(s,x)\mathrm{d}s + \int_0^t P_{s,t}(\tilde{b}\cdot\nabla\tilde{u})(s,x)\mathrm{d}s \tag{3.5}$$

$$+ P_{0,t}\phi(x), \tag{3.6}$$

where $\mathscr{L}^{\alpha}_{\tilde{\sigma}_0}$ is a infinitesimal generator of some process introduced in the crucial lemma.

- Next step is a highlight point. We operator the block operator Δ_j on both sides and only look at the point zero:

$$\Delta_j \tilde{u}(t,0) = \int_0^t \Delta_j P_{s,t}\left(\mathscr{L}^{\alpha}_{\tilde{\sigma}} - \mathscr{L}^{\alpha}_{\tilde{\sigma}_0}\right)\tilde{u}(s,0)\mathrm{d}s + \int_0^t \Delta_j P_{s,t}(\tilde{b}\cdot\nabla\tilde{u})(s,0)\mathrm{d}s$$
$$+ \Delta_j P_{0,t}\phi(0).$$

- Notice that $\Delta_j u(t,\theta_t) = \Delta_j \tilde{u}(t,0)$. We take the supremum of the initial point of the θ_t and get the estimate of $\|\Delta_j u(t)\|_\infty$. Then by taking sipremum of j, we have for some $\vartheta > -1$ and any $\gamma_1 \in [0,\alpha)$:

$$\|u(t)\|_{B^{\gamma_1}_{\infty,\infty}} \lesssim \int_0^t (t-s)^{\vartheta}\|u(s)\|_{B^{\gamma_1}_{\infty,\infty}}\mathrm{d}s + t^{-\frac{1}{\alpha}\left(\frac{d}{p}-\gamma_2+\gamma_1\right)}\|\phi\|_{B^{\gamma_2}_{p,\infty}}.$$

- Notice that a highlight point here is that we turn the convolution $P_{s,t}f$ into an inner product $\langle p_{s,t}, f\rangle$. Therefore, we use our crucial lemma and get the regularity of the space.

Volterra-type Gronwall inequality

Lemma 8 (Volterra-type Gronwall inequality)

Assume $A > 0$. For any $\theta, \vartheta > -1$ and $T > 0$, there exists a constant $C = C(A, \theta, \vartheta, T) \geqslant 0$ such that if locally integrable functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy

$$f(t) \leqslant A \int_0^t (t-s)^\theta f(s) \mathrm{d}s + At^\vartheta, \quad t \in (0, T],$$

then

$$f(t) \leqslant Ct^\vartheta, \quad t \in (0, T].$$

- When $\frac{d}{p} - \gamma_2 + \gamma_1 < \alpha$, $t^{-\frac{1}{\alpha}\left(\frac{d}{p} - \gamma_2 + \gamma_1\right)}$ is a local integral function on $[0, T]$. We obtain main result for $\gamma_1 \in [0, \alpha)$ and $\frac{d}{p} - \gamma_2 < \alpha - \gamma_1$.
- To lift the limitation of γ_1 from $[0, \alpha)$ to $[0, \alpha + \alpha \wedge 1)$, we need a lift theorem by the semigroup property of Feller process.
- The proof can be found in [1].

[1] X. Zhang, Stochastic Volterra equations in Banach spaces and stochastic partial differential equation. *J.Funct. Anal., 258 (2010), 1361-1425.*

Lift lemma

Lemma 9

Assume one of the following conditions holds,

- $\alpha \in (0,2)$, $b \equiv 0$ *and let* $\beta = 1$.
- $\alpha \in (\frac{1}{2}, 2)$ *and condition* $(\mathbf{H}^b_\beta)$ *holds with* $\beta \in ((1-\alpha) \vee 0, \alpha \wedge 1)$.

Under condition $(\mathbf{H}^\sigma_\mu)$, *for any*

$$\gamma \in (\alpha, \alpha + \alpha \wedge \beta), \quad \delta \in [0, \alpha),$$

there is a constant C_T *such that for all* $\phi \in C_0^\infty(\mathbb{R}^d)$ *and all* $t \in (0,T]$,

$$\|P_t^{\sigma,b}\phi\|_{B^\gamma_{\infty,\infty}} \leqslant C_T t^{-\frac{\delta}{\alpha}} \|\phi\|_{B^{\gamma-\delta}_{p,\infty}}. \tag{3.7}$$

- Notice that $P_t^{\sigma,b}\phi = P_{\frac{t}{2}}^{\sigma,b} P_{\frac{t}{2}}^{\sigma,b}\phi$ and $(\alpha, \alpha + \alpha \wedge \beta) - \alpha \subset (0, \alpha)$, by this C-K property, we obtain the main result.

Characteristic line

- Let θ_t^y be a solution of following ODE

$$\begin{cases} \mathrm{d}\theta_t^y = -b(\theta_t^y), \\ \theta_0^y = y, \end{cases}$$

for $t \in [0, T]$ and $y \in \mathbb{R}^d$.

Remark 10

Under the condition $\boldsymbol{H}_\beta^b$, there is a constant C such that for any $|x-y| \geqslant 1$,

$$|b(x) - b(y)| \leqslant C|x-y|,$$

which implies that θ_t^y would never blow up. See Wang-Zhang[1].

[1] Degenerate SDE with Hölder-Dini drift and non-Lipschitz coefficient. SIAM J. Math. Anal. 48 (2016), 2189–2226.

Perturbation

- Define $\Theta_t^y g(x) := g(x + \theta_t^y)$. Then $\Theta_t^y u$ satisfies a new PDE

$$\begin{cases} \partial_t \Theta_t^y u(t,x) = \mathscr{L}_0^\alpha \Theta_t^y u(t,x) + \tilde{\mathscr{L}}^\alpha \Theta_t^y u(t,x) + \tilde{b}(x) \cdot \nabla \Theta_t^y u(t,x), \\ \Theta_t^y u(0,x) = \phi(x+y), \end{cases} \tag{3.8}$$

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- where $\tilde{b}(x) = \Theta_t^y b(x) - \Theta_t^y b(0)$,

$$\mathscr{L}_0^\alpha g(x) = \int_{\mathbb{R}^d} \Big(g(x+\sigma(\theta_t^y, z)) - g(x) - \mathbb{1}_{\alpha\geqslant 1}\sigma(\theta_t^y, z)\cdot \nabla g(x) \Big)\nu(\mathrm{d}z),$$

$$\tilde{\mathscr{L}}^\alpha g(x) = \int_{\mathbb{R}^d} \mathscr{D}_z^y g(x)\nu(\mathrm{d}z)$$

$$:= \int_{\mathbb{R}^d} \Big(g(x+\sigma(x+\theta_t^y, z)) - g(x+\sigma(\theta_t^y, z)) - \mathbb{1}_{\alpha\geqslant 1}\tilde{\sigma}(x,z)\cdot \nabla g(x) \Big)\nu(\mathrm{d}z),$$

with $\tilde{\sigma}(x,z) = \sigma(x+\theta_t^y, z) - \sigma(\theta_t^y, z)$.

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with $\tilde{\sigma}(x,z) = \sigma(x+\theta_t^y,z) - \sigma(\theta_t^y,z)$.

- Notice that there is a constant C such that $|\tilde{b}(x)| \leqslant C|x|^\beta \wedge |x|$ and

$$\tilde{\sigma}(0,z) = 0, \quad |\tilde{\sigma}(x,z)| \leqslant c_0|x||z|, \quad |\nabla_x\tilde{\sigma}(x,z)| \leqslant c_0|z|.$$

- Notice that $\mathscr{L}_0^\alpha$ is the infinitesimal generation of the process

$$L_{s,t}^0 = \int_s^t \int_{\mathbb{R}^d} \sigma(\theta_r^y, z)\tilde{N}(\mathrm{d}r, \mathrm{d}z).$$

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- Since the constant c_0 in condition $\mathbf{H}_\mu^\sigma$ is independent with x and z, we drop the coefficient y and denote $\sigma(r,z) := \sigma(\theta_r^y, z)$.

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- We denote by $p_{s,t}(x)$ the transition probability of $L_{s,t}^0$, then crucial lemma is available for $p_{s,t}$. By the Duhamel's formula,

$$\begin{aligned}\Theta_t^y u(t, w) = &\int_0^t \int_{\mathbb{R}^d} p_{s,t}(w-x)\tilde{\mathscr{L}}^\alpha \Theta_t^y u(s,x)\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}^d} p_{s,t}(w-x)\tilde{b}(x)\cdot\nabla\Theta_t^y u(s,x)\mathrm{d}x\mathrm{d}s \\ &+ \int_{\mathbb{R}^d} p_{0,t}(w-x)\phi(x+y)\mathrm{d}x.\end{aligned}$$

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- We operate the block operator Δ_j on both sides and let $w=0$,

$$\begin{aligned}&\Delta_j u(t,\theta_t^y) = \Delta_j \Theta_t^y u(t,0) = \int_0^t \int_{\mathbb{R}^d} \Delta_j p_{s,t}(-x) \tilde{\mathscr{L}}^\alpha \Theta_t^y u(s,x) \mathrm{d}x\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}^d} \Delta_j p_{s,t}(-x) \tilde{b}(x) \cdot \nabla \Theta_t^y u(s,x) \mathrm{d}x\mathrm{d}s + \int_{\mathbb{R}^d} \Delta_j p_{0,t}(-x) \phi(x+y) \mathrm{d}x, \\ &:= \mathscr{I}_1^j + \mathscr{I}_2^j + \mathscr{I}_3^j.\end{aligned} \tag{3.9}$$

Estimate for $\mathscr{I}_3^j$

- $(\tilde{\Delta}_j \Delta_j = \Delta_j$ and Δ_j is symmetric$)\Rightarrow$

$$\mathscr{I}_3^j = \int_{\mathbb{R}^d} \Delta_j p_{0,t}(-x)\phi(x+y)dx = \int_{\mathbb{R}^d} \Delta_j p_{0,t}(-x)\tilde{\Delta}_j \phi(x+y)dx.$$

- (Hölder inequlity)$\Rightarrow$

$$|\mathscr{I}_3^j| \leqslant \int_{\mathbb{R}^d} |\Delta_j p_{0,t}(-x)||\tilde{\Delta}_j \phi(x+y)|dx \leqslant \|\Delta_j p_{0,t}\|_{L^q} \|\tilde{\Delta}_j \phi\|_{L^p},$$

  where $\frac{1}{p} + \frac{1}{q} = 1.$

- (Definition of Besov space and crucial lemma 7)$\Rightarrow$

$$|\mathscr{I}_3^j| \leqslant 2^{-\gamma_2 j} \|\Delta_j p_{0,t}\|_{L^q} \|\phi\|_{B^{\gamma_2}_{p,\infty}} \lesssim 2^{-\gamma_1 j} t^{-\frac{1}{\alpha}(\frac{d}{p} - \gamma_2 + \gamma_1)} \|\phi\|_{B^{\gamma_2}_{p,\infty}}.$$

- Notice that $\frac{d}{p} - \gamma_2 + \gamma_1 \geqslant 0$, which is $\gamma_2 \leqslant \frac{d}{p} + \gamma_1$.

Estimate for $\mathscr{I}_2^j$

- Define function $\chi \in C_0^\infty$ with

$$\chi(x) = \begin{cases} 1 & \text{when } |x| < \frac{1}{2} \\ 0 & \text{when } |x| > 1. \end{cases}$$

Lemma 11

Under condition $\boldsymbol{H}^b_\beta$, function $b_z(x) := \chi(x)\Big(b(x+z) - b(z)\Big) \in C^\beta(\mathbb{R}^d)$. There is a constant C such that all $z \in \mathbb{R}^d$

$$\|b_z\|_{C^\beta} \leqslant C.$$

- By Lemma11 and the fact that

$$\|f\|_{C^\beta(\mathbb{R}^d)} \lesssim \sup_{z\in\mathbb{R}^d} \|f\|_{C^\beta(B(z,1))},$$

we assume $b \in C^\beta$ and have a commutator estimate:

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Lemma 12 (Chen-Zhang-Zhao 2017)

For $\beta \in (0,1)$ and $\theta \in (-\beta, 0]$, there is a constant C such that

$$\|[\Delta_j, f]g\|_\infty \leqslant C2^{-j(\beta+\theta)}\|f\|_{C^\beta}\|g\|_{B^\theta_\infty},$$

where $[\Delta_j, f]g := \Delta_j fg - f\Delta_j g$.

Estimate for $\mathscr{I}_2^j$

Lemma 13

Assume $\alpha \in (\frac{1}{2}, 2)$. *Under condition* $\boldsymbol{H}_\beta^b$ *with* $\beta \in ((1-\alpha) \vee 0, \alpha \wedge 1)$. *For any* $\gamma_1 \in (0, \alpha)$ *and* $T > 0$, *there is a constant* C *such that for all* $t \in (0, T]$, $j \in \mathbb{N}_0$ *and all classical solution* u,

$$|\mathscr{I}_2^j| \leqslant C 2^{-\gamma_1} \int_0^t (t-s)^{-\frac{2\gamma_1+\beta-1}{\alpha}} \|u(s)\|_{C^{\gamma_1}} \, \mathrm{d}s.$$

Estimate for $\mathscr{I}_2^j$

Lemma 13

Assume $\alpha \in (\frac{1}{2}, 2)$. *Under condition* $\boldsymbol{H}_\beta^b$ *with* $\beta \in ((1-\alpha) \vee 0, \alpha \wedge 1)$. *For any* $\gamma_1 \in (0, \alpha)$ *and* $T > 0$, *there is a constant* C *such that for all* $t \in (0, T]$, $j \in \mathbb{N}_0$ *and all classical solution* u,

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Proof.

Notice that

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^d} \Delta_j p_{s,t}(-x) \tilde{b}(x) \cdot \nabla \Theta_t^y u(s,x) \mathrm{d}x \mathrm{d}s \\ =& \int_0^t \int_{\mathbb{R}^d} \Delta_j p_{s,t}(-x) [\tilde{\Delta}_j, \tilde{b}(x)] \nabla \Theta_t^y u(s,x) \mathrm{d}x \mathrm{d}s \\ +& \int_0^t \int_{\mathbb{R}^d} \Delta_j p_{s,t}(-x) \tilde{b}(x) \cdot \tilde{\Delta}_j \nabla \Theta_t^y u(s,x) \mathrm{d}x \mathrm{d}s. \end{aligned}$$

By crucial lemma and commutator estimate, we complete the proof. □

Estimate for $\mathscr{I}_1^j$

- Recall that

$$\mathscr{D}_z^y f(x) = f(x+\sigma(x+\theta_t^y, z)) - f(x+\sigma(\theta_t^y, z)) - \mathbb{1}_{\alpha\geqslant 1}\tilde{\sigma}(x,z)\cdot\nabla f(x).$$

- Define

$$\mu_\theta(h) := \int_{\mathbb{R}^d} (1\wedge|x|)^\theta |h(x)|dx \quad \text{and} \quad \langle f, g\rangle := \int_{\mathbb{R}^d} f(x)g(x)dx.$$

Estimate for $\mathscr{I}_1^j$

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Lemma 14

For any $\theta\in[0,1]$, *there exists a constant* $C = C(d,\theta) > 0$ *such that for all* $|z|\leqslant \frac{1}{2c_0}$, $f\in C^\theta$ *and* $g\in C^2$

$$|\langle \mathscr{D}_z^y f, g\rangle| \leqslant C|z|^\theta \|f\|_\infty \left[\mu_0(|g|) + \mu_\theta(|\nabla g|)^\theta \mu_\theta(|g|)^{1-\theta}\right]$$

when $\alpha < 1$ *and*

$$|\langle \mathscr{D}_z^y f, g\rangle| \leqslant C|z|^{1+\theta} \|f\|_{\mathbf{C}^\theta} \left[\mu_0(|g|) + \mu_1(|\nabla g|) + \mu_{1+\theta}(|\nabla^2 g|)^\theta \mu_{1+\theta}(|\nabla g|)^{1-\theta}\right]$$

when $\alpha\geqslant 1$.

The key point of the proof

- For simplicity, we assume $\alpha < 1$ and $\phi_z(x) = \sigma(x + \theta_t^y, z)$. Rewrite

$$\mathscr{D}_z f(x) := \mathscr{D}_z^y f(x) = f(x + \phi_z(x)) - f(x + \phi_z(0)).$$

- We can let $\bar{f}(x) = f(x + \phi_z(0))$. Their C^θ norms are the same. Therefore we assume that $\phi_z(0) = 0$ and there is a constant such that $|\phi_z(x)| \leqslant C(|x| \wedge 1)|z|$.

The key point of the proof

- For simplicity, we assume $\alpha < 1$ and $\phi_z(x) = \sigma(x + \theta_t^y, z)$. Rewrite

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- We can let $\bar{f}(x) = f(x + \phi_z(0))$. Their C^θ norms are the same. Therefore we assume that $\phi_z(0) = 0$ and there is a constant such that $|\phi_z(x)| \leqslant C(|x| \wedge 1)|z|$.
- Let $\Gamma_z(x) = x + \phi_z(x)$. By change of variable, we have

$$\langle \mathscr{D}_z f, g \rangle = \langle f, \mathscr{D}_z^* g \rangle,$$

where

$$\mathscr{D}_z^* g(x) = \det(\nabla_x \Gamma_z^{-1}(x)) g(\Gamma_z^{-1}(x)) - g(x).$$

- Noticing that

$$|\det(\nabla_x \Gamma_z^{-1}(x)) - 1| \leqslant |z|, \quad \text{and} \quad |\Gamma_z^{-1}(x) - x| \leqslant CC(|x| \wedge 1)|z|,$$

we complete the proof.

Lemma 15

Let $\varepsilon \in (0, \alpha \wedge 1)$ *and* $\theta \in ((\alpha - 1) \vee 0, \alpha \wedge 1)$.*For any* $\gamma \in (0, \alpha - \varepsilon)$, *there is a constant* $C > 0$ *such that for all* $j \in \mathbb{N}_0$ *and* $t \in (0, T]$,

$$|\mathscr{I}_1^j| \leqslant C 2^{-\gamma j} \int_0^t (t-s)^{-(\gamma+\varepsilon)/\alpha} \|u(s)\|_{C^\theta} \mathrm{d}s.$$

- Recall

$$\mathscr{I}_1^j = \int_0^t \int_{\mathbb{R}^d} \Delta_j p_{s,t}(-x) \tilde{\mathscr{L}}^\alpha \Theta_t^y u(s,x) \mathrm{d}x \mathrm{d}s.$$

Proof.

Let $\delta = \frac{\kappa}{c_0}$.We only prove the estimate for $\alpha \in (1,2)$. The case $\alpha \in (0,1]$ is similar and easier.Since the time variable and y does not play any essential role, below we drop the time variable and Θ_t^y for simplicity of notations. By definition we can make the following decomposition:

$$\tilde{\mathscr{L}}^\alpha u = \mathscr{A}_\delta u + \bar{\mathscr{A}_\delta} u,$$

where

$$\mathscr{A}_\delta u(x) = \int_{|z| \leqslant \delta} \mathscr{D}_z u(x) \nu(\mathrm{d}z) \quad \text{and} \quad \bar{\mathscr{A}_\delta} u(x) = \int_{|z| > \delta} \mathscr{D}_z u(x) \nu(\mathrm{d}z).$$

□

Proof.

$$\mathscr{I}_1^j = \int_0^t \langle \Delta_j p_{s,t}, \mathscr{A}_\delta u \rangle ds + \int_0^t \langle \Delta_j p_{s,t}, \bar{\mathscr{A}_\delta} u \rangle ds.$$

By Lemma14, we have

$$|\langle \Delta_j p_{s,t}, \mathscr{A}_\delta u \rangle| \leqslant C \int_{|z|\leqslant\delta} |z|^{1+\theta} \nu(\mathrm{d}z) \|u(s)\|_{\mathbf{C}^\theta} \mathscr{B}(s,t),$$

where

$$\mathscr{B}(s,t) = \left| \sum_{i=0}^{1} \mu_i(|\nabla^i \Delta_j p_{s,t}|) + \mu_{1+\theta}(|\nabla^2 \Delta_j p_{s,t}|)^\theta \mu_{1+\theta}(|\nabla \Delta_j p_{s,t}|)^{1-\theta} \right|.$$

Let $\alpha < 1+\theta < \alpha + \frac{\varepsilon}{2}$. By crucial lemma, we obtain that

$$|\int_0^t \langle \Delta_j p_{s,t}, \mathscr{A}_\delta u \rangle \mathrm{d}s| \leqslant C \int_0^t \|u(s)\|_{\mathbf{C}^\theta} \mathscr{B}(s,t) ds$$

$$\lesssim 2^{-\gamma j} \int_0^t (t-s)^{-\frac{\gamma}{\alpha}} \|u(s)\|_{C^\theta} \mathrm{d}s + 2^{-(\gamma-\varepsilon)j} \int_0^t (t-s)^{-\frac{\gamma}{\alpha}} \|u(s)\|_{C^\theta} \mathrm{d}s,$$

where

$$\int_0^t \mu_{1+\theta}(|\nabla^j \Delta_j p_{s,t}|) \|u(s)\|_{C^\theta} \mathrm{d}s \lesssim \int_0^t \int_{\mathbb{R}^d} |x|^{\alpha-\frac{\varepsilon}{2}} |\nabla^j \Delta_j p_{s,t}(x)| \|u(s)\|_{C^\theta} \mathrm{d}s.$$ □

Proof.

so

$$\int_0^t \mu_{1+\theta}(|\nabla^2\Delta_j p_{s,t}|)^\theta \mu_{1+\theta}(|\nabla\Delta_j p_{s,t}|)^{1-\theta}\|u(s)\|_{C^\theta}\mathrm{d}s$$

$$\lesssim 2^{-(\alpha-\frac{\varepsilon}{2}-1-\theta+\gamma)j}\int_0^t (t-s)^{-\frac{\gamma}{\alpha}}\|u(s)\|_{C^\theta}\mathrm{d}s \leqslant 2^{-\gamma j}\int_0^t (t-s)^{-\frac{\gamma}{\alpha}}\|u(s)\|_{C^\theta}\mathrm{d}s.$$

For $\bar{\mathscr{A}}_\delta u$, by Fubini's theorem and the integration by parts, we have

$$|\langle \Delta_j p_{s,t}, \bar{\mathscr{A}}_\delta u\rangle| \leqslant \int_{|z|>\delta}\int_{\mathbb{R}^d} |\Delta_j p_{s,t}(x)||u(x+\sigma(x,z)) - u(x+\sigma(0,z))|\mathrm{d}x\mathrm{d}z$$

$$+\int_{|z|>\delta}\int_{\mathbb{R}^d}\left|\Big(\Delta_j p_{s,t}(x)\mathrm{div}_x\sigma(x,z) + (\sigma(x,z)-\sigma(0,z))\cdot\nabla\Delta_j p_{s,t}(x)\Big)u(x)\right|\mathrm{d}x\mathrm{d}z$$

$$\leqslant \|u(s)\|_\infty\Big(\mu_0(|\Delta_j p_{s,t}|) + \mu_1(|\nabla\Delta_j p_{s,t}|)\int_{|z|>\delta}|z|\mathrm{d}z\Big).$$

By crucial lemma again, we obtain that

$$\int_0^t |\langle \Delta_j p_{s,t}, \bar{\mathscr{A}}_\delta u\rangle|\mathrm{d}s \lesssim 2^{-\gamma j}\int_0^t (t-s)^{\frac{\gamma}{\alpha}}\|u(s)\|_\infty\mathrm{d}s.$$

□

Future works

- We prove that the solution of SDE driven by cylindrical Lévy process has a density in Sobolev space $H^{s,r}$ with

$$s < \alpha - (\alpha - 1) \vee (1 - \beta) \quad \text{and} \quad r < \frac{d}{d - \alpha + s + \beta - 1},$$

  but this result does not imply that this density is continuous. So how to improve the index s and how to make r greater are interesting.
- In our work, we only consider the strong Feller property, which only depend on the distribution of X_t^x. Moreover, the continuous property of σ is enough to guarantee the existence of weak solution. So how to drop the assumption that σ is Lipschitz is another interesting question.

Thanks for your attention!