Quantitative approximation of kinetic SDEs: from discrete to continuum

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Joint work with Khoa Lê and Chengcheng Ling

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Kinetic models

- Newton's Second Law of Motion:

$$\dot{V}_t = \ddot{X}_t = F(X_t, V_t)$$

- The distribution $\mu_t = \mu_t(\mathrm{d}x, \mathrm{d}v) := \mathbb{P} \circ (X_t, V_t)^{-1}$ solves the following kinetic equation:

$$\partial_t \mu_t + v \cdot \nabla_x \mu_t + \mathrm{div}_v(F \mu_t) = 0$$

- Maxwell 1867; Boltzmann 1872; Vlasov 1938; Landau 1936;

  Chapman-Enskog method 1970:
  Mesoscopic (Boltzmann equation) and **macroscopic** (Navier-Stokes equations),
  ...., Villani, Lions, Golse, Bouchut, Imbert, Mouhot, Silvestre, Guo, Mourrat, ...

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- **Microscopic** (Kinetic SDEs) and **mesoscopic** (Boltzmann equation):
  Tanaka 1978, PTRF., Mischler-Mouhot 2013, Invent.,...
- (Difficulty): degenerate parabolic PDE;
  the transfer of regularity between x and v.

Kinetic SDEs and scaling

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$$\begin{cases} \mathrm{d}X_t = V_t \mathrm{d}t, \\ \mathrm{d}V_t = b(X_t, V_t)\mathrm{d}t + \mathrm{d}B_t. \end{cases}$$

- Scaling ($b = 0$):

$$(X_{\varepsilon^2 t}, V_{\varepsilon^2 t}) = (\int_0^{\varepsilon^2 t} B_s \mathrm{d}s, B_{\varepsilon^2 t}) = (\int_0^t \varepsilon^2 B_{\varepsilon^2 s'} \mathrm{d}s', B_{\varepsilon^2 t})$$

$$\overset{(d)}{=} (\varepsilon^3 \int_0^t B_{s'} \mathrm{d}s', \varepsilon B_t) = (\varepsilon^3 X_t, \varepsilon V_t).$$

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- Heat kernel estimate: (Kolmogorov, 1934): $p_t(x, v) \sim (X_t, V_t)$ with $b = 0$,

$$p_t(x, v) = (4\pi t^4/3)^{-d/2} \exp\left(-(3|x|^2 + |3x - 2tv|^2)/(4t^3)\right).$$

  - $p_t(x, v) = t^{-2d} p_1(t^{-3/2}x, t^{-1/2}v)$.
- (Chaudru-Menozzi-Zhang, 2023. Bull. Sci. Math.), ...

Well-posedness

- Kinetic SDEs: $V = \dot{X}$,

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- Weak well-posedness [Weak existence + Uniqueness in law]
  - (Chaudru de Raynal-Menozzi, 2021. TAMS) $b \in L_t^q L_{x,v}^p$, $\boxed{\frac{2}{q} + \frac{4d}{p} < 1}$.
  - (Ren-Zhang, 2024. Bernoulli) Kato class, $b \in L_t^q L_x^{p_x} L_v^{p_v}$, $\boxed{\frac{2}{q} + \frac{3d}{p_x} + \frac{d}{p_v} < 1}$.

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  - (Chaudru de Raynal, 2017. AIHP) $b \in C_x^{\frac{2}{3}+} \cap C_v^{0+}$.
  - (Wang-Zhang, 2016. SIAM) Dini continuous.
  - (Fedrizzi-Flandoli-Priola-Vovelle, 2017. EJP) $(1-\Delta_x)^{\frac{1}{3}+} b \in L_{x,v}^p$, $p > 6d$.
  - (Zhang, 2018. Sci. China) $(1-\Delta_x)^{\frac{1}{3}} b \in L_{t,x,v}^p$, $\boxed{\frac{2}{p} + \frac{3d}{p} + \frac{d}{p} < 1}$.

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  - (Our results) There is some $\beta > 0$,

$$(1 + (1-\Delta_x)^{\frac{1}{3}} - \Delta_v)^{\frac{\beta}{2}} (1-\Delta_x)^{\frac{1}{3}} b \in L^{p_x}_x L^{p_v}_v, \ \boxed{\frac{3d}{p_x} + \frac{d}{p_v} < 1}. \quad (H_\beta)$$

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- Distribution drift: (H.-Zhang-Zhu-Zhu, 2024. AOP), (Issoglio-Pagliarani-Russo-Trevisani, 2024.) ...
- Lévy processes cases: (Chen-Zhang, 2018. JMPA), (H.-Wu-Zhang, 2020. JMPA), (Marino-Menozzi, 2023. SPA)...

Euler-Maruyama scheme

- Let $n \in \mathbb{N}$ and $k_n(t) := [nt]/n$.

$$\begin{cases} \mathrm{d}X_t^n = V_t^n \mathrm{d}t, \\ \mathrm{d}V_t^n = b(X_{k_n(t)}^n, V_{k_n(t)}^n)\mathrm{d}t + \mathrm{d}B_t. \end{cases}$$

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- (Lemaire-Menozzi, 2010. EJP) b is bounded:
  Weak convergence: $\mathbb{P} \circ (X^n, V^n)^{-1} \to \mathbb{P} \circ (X, V)^{-1}$, as $n \to \infty$.

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- (Leobacher-Szölgyenyi, 2018. Numer. Math.)
  b is piecewise continuous and bounded:

$$\mathbb{E}\left(\sup_{t\in[0,1]} |(X_t, V_t) - (X_t^n, V_t^n)|\right) \lesssim n^{-\frac{1}{4}+}.$$

Tamed Euler-Maruyama scheme with time transport

- Let $\boxed{\Gamma_t f(x,v) := f(x+tv, v)}$ and $\boxed{b_n := b * \phi_n}$, where for a smooth probability density function ϕ and $\theta > 0$,

$$\phi_n(x,v) := n^{4d\theta}\phi(n^{3\theta}x, n^{\theta}v).$$

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- Benefits of adding $\Gamma_{t-k_n(t)}$:
  Let $X_t := \int_0^t B_s \mathrm{d}s$ and $f(x,v) = f(x)$ be a Lipschitz function.
  - (Without $\Gamma_{t-k_n(t)}$) : $\mathbb{E}|f(X_t) - f(X_{k_n(t)})| \leq \|f\|_{Lip} \int_{k_n(t)}^t s^{1/2}\mathrm{d}s \lesssim n^{-1}$.

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  - (With $\Gamma_{t-k_n(t)}$) :

$$\begin{aligned} \mathbb{E}|f(X_t) - \Gamma_{t-k_n(t)} f(X_{k_n(t)})| &\leq \|f\|_{Lip} \int_{k_n(t)}^t \mathbb{E}|W_s - W_{k_n(s)}|\mathrm{d}s \\ &\lesssim \int_{k_n(t)}^t (s-k_n(s))^{1/2}\mathrm{d}s \lesssim n^{-3/2}. \end{aligned}$$

Main results

- Assume that $b \in L_x^{p_x} L_v^{p_v}$ with
$$\alpha := \frac{3d}{p_x} + \frac{d}{p_v} < 1.$$
Let $\theta \in (0, \frac{1}{2\alpha})$.

Theorem 1

- *Weak convergence:*
$$\int_0^1 \|\mathbb{P} \circ (X_t, V_t)^{-1} - \mathbb{P} \circ (X_t^n, V_t^n)^{-1}\|_{\text{var}} \mathrm{d}t \lesssim n^{-\frac{1}{2}} + n^{-\theta}.$$
- *Strong convergence: Under the condition* (H_β),
$$\mathbb{E}\left[\sup_{t\in[0,1]} |(X_t, V_t) - (X_t^n, V_t^n)|\right] \lesssim n^{-\frac{1+\beta/3}{2}+\varepsilon} + n^{-(\beta+1-\alpha)\theta+\varepsilon}, \quad \forall \varepsilon > 0.$$

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- When b is bounded, we can replace b_n with b and let $\theta = +\infty$.
- The key point in the proof: Littlewood-Paley's type estimate for the heat kernel.
- Difficulty: $p_t * p_s \neq p_{t+s}$; $p_t * (\Gamma_s p_s) = \Gamma_s p_{t+s}$.

Remarks:

(i) We don't need any continuous assumption on b.

(ii) In contrast to (Jourdain-Menozzi, 2024. AAP) for $b \in L^p$ and

$$dX_t = b(X_t)dt + dW_t, \tag{1}$$

$$\|\mathbb{P} \circ (X_t)^{-1} - \mathbb{P} \circ (X_t^n)^{-1}\|_{\text{var}} \lesssim n^{-\frac{1-d/p}{2}},$$

our weak convergence rate is $1/2$ when $\theta \in [1/2, 1/(2\alpha)$, which is independent of (p_x, p_v).

(iii) The strong convergence rate is $1/2$ when $\alpha < 1/2$ and $\theta \in (1/(2(1-\alpha)), 1/(2\alpha))$, which coincides the results in (Lê-Ling, 2021) for SDE (1).

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Open questions:

(i) Is the convergence rate $1/2$ optimal (especially for the weak case)? (Ellinger-Müller-Gronbach-Yaroslavtseva 2024)

(ii) When $b(x,v) = b(x)$, would it be possible to improve the convergence rate from $\frac{1}{2}$ to $\frac{3/2}{2}$(=3/4)?

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