SDEs with supercritical distribution drifts

Zimo Hao

Bielefeld University

Joint work with Xicheng Zhang

12 March 2024

Non-local operators, probability and singularities

Table of contents

Overview

- Consider the following SDE

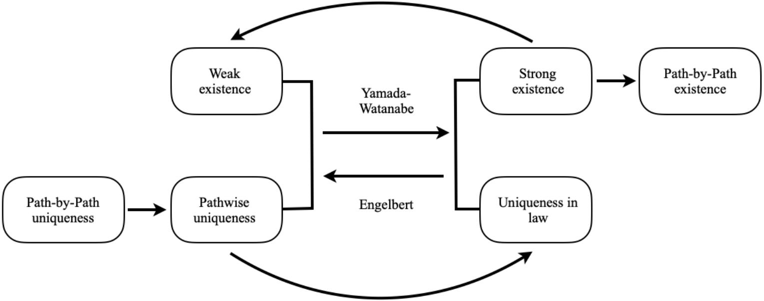
$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t, \tag{1}$$

where $(W_t)_{t \geq 0}$ is a standard d-dimensional Brownian motion and $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a measurable function.

  - Weak solution: $(\Omega, \mathscr{F}, \mathbf{P}, (\mathscr{F}_s)_{s \geq 0}, W, X)$;
  - Strong solution: $(\Omega, \mathscr{F}, \mathbf{P}, (\mathscr{F}_s)_{s \geq 0}, W) \Rightarrow X = \Phi(X_0, W)$;
  - Maringale solution: $\mathbb{P} \in \mathscr{P}(\mathcal{C}_T)$, for all $f \in \mathbf{C}^2(\mathbb{R}^d)$

$$f(\omega_t) - f(\omega_0) - \int_0^t (\Delta + b \cdot \nabla) f(\omega_s)\mathrm{d}s \quad \text{is a } \mathbb{P}\text{-martingale;}$$

  - Path-by-path solution: for any path $t \to W_t(\omega)$, the solution solves the ODE (1).
  - Uniqueness in law; Pathwise uniqueness; Path-by-path uniqueness.
- Regularization by noise.



- ▷ (Stroock-Varadhan): Weak solution $\iff$ Martingale solution;
- ▷ (Barlow): Uniqueness in law $\not\Rightarrow$ Existence of strong solution.
- ▷ (Shaposhnikov-Wresch, Anzeletti): Many counterexamples.

- ▷ Yamada, T. and Watanabe, S. (1971). On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.*
- ▷ Engelbert, H. J. (1991). On the theorem of T. Yamada and S. Watanabe. *Stochastics Stochastics Rep.*
- ▷ Stroock, D. W. and Varadhan, S. S. R. Multidimensional diffusion processes, volume 233 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. *Springer-Verlag, Berlin*, 1979.
- ▷ Barlow, M. T. (1982). One-dimensional stochastic differential equations with no strong solution. *J. London Math. Soc.*
- ▷ Shaposhnikov, A. and Wresch, L. (2022). Pathwise vs. path-by-path uniqueness. *Ann. Inst. Henri Poincaré Probab. Stat.*
- ▷ Anzeletti, L. (2022). Comparison of classical and path-by-path solutions to SDEs. arXiv:2204.07866.

SDEs and PDEs

- Consider the following SDE:

$$X_{s,t}(x) = x + \int_s^t b(r, X_{s,r}(x))\mathrm{d}r + \sqrt{2}(W_t - W_s);$$

- Forward Fokker-Planck equation (FPE):

$$\partial_t \mu_{s,t} = \Delta \mu_{s,t} - \mathrm{div}(b(t)\mu_{s,t}), \quad \mu_{s,s} = \delta_x;$$

- Backward Fokker-Planck-Kolmogorov equation (BKE):

$$\partial_s u_{s,t} + \Delta u_{s,t} + b(s) \cdot \nabla u_{s,t} + f = 0, \quad u_{t,t} = \varphi.$$

SDEs and PDEs

$$u_{s,t}(x) = \mathbb{E}\varphi(X_{s,t}(x)) + \mathbb{E}\int_s^t f(r, X_{s,r})\mathrm{d}r$$

↑ Itô's formula to $r \to u_{r,t}(X_{s,r}(x))$

$$X_{s,t}(x)$$

↓ Itô's formula to $r \to \phi(X_{s,r}(x))$ for any $\phi \in C_b^2$

$\mathbb{P} \circ (X_{s,t}(x))^{-1}$ satisfies (FPE)

What can we say if b is not a function?

- Brox diffusion (white noise); Other noises.
- $b = \nabla U$ with some Hölder potential;
- (Weak solution): $X_t = X_0 + A_t^b + W_t$, where

$$A_t^b := \lim_{n\to\infty} \int_0^t b_n(s, X_s) \mathrm{d}s \quad \text{exists.}$$

What can we say if b is not a function?

- Brox diffusion (white noise); Other noises.
- $b = \nabla U$ with some Hölder potential;
- (Weak solution): $X_t = X_0 + A_t^b + W_t$, where

$$A_t^b := \lim_{n\to\infty} \int_0^t b_n(s, X_s)\mathrm{d}s \quad \text{exists.}$$

- (Martingale solution):
  - For any $f \in \mathbf{C}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, consider the related BKE

$$\partial_t u + \Delta u + b \cdot \nabla u + f = 0, \quad u(T) = 0.$$

    We call $\mathbb{P} \in \mathscr{P}(\mathcal{C}_T)$ a martingale solution if

$$u(t, \omega_t) - u(t, \omega_0) - \int_0^t f(r, \omega_r)\mathrm{d}r \quad \text{is a } \mathbb{P}\text{-martingale.}$$

  - N. Ethier and G. Kurtz. Markov Processes: Characterization and Convergence. *Wiley series in probability and mathematical statistic. Wiley*, 1986.

Scale analysis

- Let $\dot{\mathbf{H}}^{\alpha}_p$ be the homogenous Bessel potential space, where $\alpha \leq 0$ and $p \in [1, \infty]$ and suppose for some $q \in [1, \infty]$

$$b \in L^q(\mathbb{R}_+; \dot{\mathbf{H}}^{\alpha}_p),$$

and SDE (1) admits a solution denoted by X. For $\lambda > 0$, we define

$$X^{\lambda}_t := \lambda^{-1} X_{\lambda^2 t}, \quad W^{\lambda}_t := \lambda^{-1} W_{\lambda^2 t}, \quad b^{\lambda}(t, x) := \lambda b(\lambda^2 t, \lambda x).$$

- Then we have

$$\mathrm{d}X^{\lambda}_t = b^{\lambda}(t, X^{\lambda}_t)\mathrm{d}t + \sqrt{2}\mathrm{d}W^{\lambda}_t,$$

where

$$\|b^{\lambda}\|_{L^q(\mathbb{R}_+; \dot{\mathbf{H}}^{\alpha}_p)} = \lambda^{1+\alpha-\frac{d}{p}-\frac{2}{q}} \|b\|_{L^q(\mathbb{R}_+; \dot{\mathbf{H}}^{\alpha}_p)}.$$

- As $\lambda \to 0$,

Subcritical: $\frac{d}{p} + \frac{2}{q} < 1 + \alpha$;

Critical: $\frac{d}{p} + \frac{2}{q} = 1 + \alpha$;

Supercritical: $\frac{d}{p} + \frac{2}{q} > 1 + \alpha$.

A well-defined restriction on α

- Consider the related PDE:

$$\partial_t u = \Delta u + b \cdot \nabla u + f.$$

- Assume $b \in \mathbf{C}^\alpha$ with the differentiability index $\alpha < 0$.
- By the Schauder theory, u is at most in $\mathbf{C}^{2+\alpha}$.
- To make the product $b \cdot \nabla u$ meaningful, we need to stipulate that $1 + 2\alpha > 0$, which implies $\alpha > -\frac{1}{2}$.
  - (Delarue-Diel 2016) rough path & (Cannizzaro-Chouk 2018) paracontrolled calculus: $b \in \mathbf{C}^{-2/3+}$ is some Gaussian noise.
  - (Question) Arbitrary function b? $\alpha \to -1$?

Well-known results

SEU: Strong existence-uniqueness; WEU: Weak existence-uniqueness;
WE: Weak existence; EUP: Existence-uniqueness of path-by-path solution.

Value of α	Subcritical	Critical	Supercritical
$\alpha = 0$	SEU: $\mathrm{V}^{79}_{[1]}$, $\mathrm{KR}^{05}_{[2]}$, $\mathrm{Z}^{05,10}_{[3,4]}$ EUP: $\mathrm{D}^{07}_{[5]}$, $\mathrm{ALL}^{23}_{[6]}$	WEU&SEU: $\mathrm{BFGM}^{19}_{[7]}$, $\mathrm{K}^{21}_{[8]}$, $\mathrm{RZ}^{21}_{[9]}$, $\mathrm{KM}^{23}_{[10]}$	WE: $\mathrm{ZZ}^{21}_{[11]}$
$\alpha \in [-\frac{1}{2}, 0)$	WEU: $\mathrm{BC}^{01}_{[12]}$, $\mathrm{FIR}^{17}_{[13]}$, $\mathrm{ZZ}^{17}_{[14]}$	–	–
$\alpha \in [-1, -\frac{1}{2})$	–	–	–

[1] A. J. Veretennikov. *Theory Probab. Appl.* **24**. [2] N.V. Krylov and M. Röckner. *Probab. Theory Related Fields* **131**.
[3] X. Zhang. *Stochastic Process. Appl.* **115/11**. [4] X. Zhang. *Electron. J. Probab.* **16**.
[5] A. M. Davie. *Int. Math. Res. Not. IMRN* **24**. [6] L. Anzeletti, K. Lê and C. Ling. arXiv:2304.06802.
[7] L. Beck, F. Flandoli, M. Gubinelli and M. Maurelli. *Electron. J. Probab.* **24**.
[8] N. V. Krylov. *Ann. Probab.* **49**. [9] M. Röckner and G. Zhao. *Bernoulli* **29** and arXiv:2103.05803.
[10] D. Kinzebulatov and K. R. Madou. arXiv:2306.04825. [11] X. Zhang and G. Zhao. *Commun. Math. Phys.* **381**.
[12] R. F. Bass R.F. and Z. Q. Chen. *Probability Theory and Related Fields.* **121**.

[13] F. Flandoli, E. Issoglio and F. Russo. *Trans. Am. Math. Soc.* **369**. [14] X. Zhang and G. Zhao. arXiv:1710.10537.

Zvonkin's transformation- a method to kill the drift

- Consider the following BKE:

$$\partial_t\Phi + \Delta\Phi + b\cdot\nabla\Phi = 0, \quad \Phi(T,x) = x,$$

where $\Phi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$. We assume that if we can use Itô's formula to $s \to \Phi(s, X_s)$ and then

$$\mathrm{d}\Phi(t,X_t) = \sqrt{2}\nabla\Phi(t,X_t)\mathrm{d}W_t.$$

- We assume that $\Phi(t,\cdot)$ is an C^1-diffeomorphism.
- We define $(Y_t)_{t\geq 0} := (\Phi(t,X_t))_{t\geq 0}$ and note that $(Y_t)_{t\geq 0}$ satisfies the SDE without drift.

Weak well-posedness of subcritical SDEs with $\alpha \in (-1, -\frac{1}{2})$

Main results

($\mathbf{H}^{\text{sub}}$) Let $(\alpha, p, q) \in (-1, -\frac{1}{2}] \times [2, \infty)^2$ with $\frac{d}{p} + \frac{2}{q} < 1 + \alpha$. Suppose that

$$\kappa_1^b := \|b\|_{\mathbb{L}_T^q \mathbf{B}_{p,q}^{\alpha}} < \infty \quad \text{and} \quad \kappa_2^b := \|\mathrm{div} b\|_{\mathbb{L}_T^q \mathbf{B}_{p,q/(q-1)}^{-2-\alpha}} < \infty.$$

Theorem 1 (H.-Zhang 2023)

*Under the condition (**H^{sub}**), there is a unique weak solution to SDE* (1). *Moreover, $t \to A_t^b$ has finite p-variation with some $p < 2$.*

Main results

($\mathbf{H}^{\rm sub}$) Let $(\alpha, p, q) \in (-1, -\frac{1}{2}] \times [2, \infty)^2$ with $\frac{d}{p} + \frac{2}{q} < 1 + \alpha$. Suppose that

$$\kappa_1^b := \|b\|_{\mathbb{L}^q_T \mathbf{B}^{\alpha}_{p,q}} < \infty \quad \text{and} \quad \kappa_2^b := \|\mathrm{div} b\|_{\mathbb{L}^q_T \mathbf{B}^{-2-\alpha}_{p,q/(q-1)}} < \infty.$$

Theorem 1 (H.-Zhang 2023)

Under the condition **($\mathbf{H}^{\rm sub}$)**, *there is a unique weak solution to SDE* (1). *Moreover,* $t \to A^b_t$ *has finite* p*-variation with some* $p < 2$.

- Suppose that $b \in \mathbb{L}^q_T \mathbf{B}^{-1/2}_{p,1}$ with $\frac{d}{p} + \frac{2}{q} < \frac{1}{2}$. Then **($\mathbf{H}^{\rm sub}$)** holds for $\alpha = -\frac{1}{2}$. Moreover, when $\mathrm{div} b = 0$, **($\mathbf{H}^{\rm sub}$)** holds.
- For any Lipschitz function $g : \mathbb{R}^d \to \mathbb{R}$,

$$\int_0^t g(X_s) \mathrm{d}A^b_s \quad \text{is a Young integral.}$$

Example:Gaussian noises

- For given $\gamma \in (d-2, d)$, we define the Gaussian noise b by the following covariance

$$\mathbb{E}b(f)b(g) = \int_{\mathbb{R}^d} \hat{f}(\xi)\hat{g}(-\xi)|\xi|^{-\gamma}\Big(\mathbb{I}_{d\times d} - \frac{\xi \otimes \xi}{|\xi|^2}\Big)\mathrm{d}\xi.$$

- Then we have for almost surely ω

$$b(\omega, \cdot) \in \cap_{p\in[1,\infty)} \mathbf{B}^{-1+}_{p,loc}(\mathbb{R}^d) \quad \mathrm{div} b(\omega) = 0.$$

Sketch of the proof

- Consider the following BKE:

$$\partial_t u + \Delta u + b \cdot \nabla u + f = 0, \quad u(T) = 0, \quad t \in [0, T].$$
$$b \in \mathbf{C}^{\alpha}, \quad u \in \mathbf{C}^{2+\alpha}.$$

- We define $b \cdot \nabla u := b \odot \nabla u + \mathrm{div} b \circ u + \mathrm{div} b \prec u$ where

$$\boxed{b \odot \nabla u := \mathrm{div}(b \prec u + b \circ u) + b \succ \nabla u.}$$

- The paraproduct implies that

$$\|\mathrm{div} b \circ u + \mathrm{div} b \prec u\|_{\alpha} \lesssim \|\mathrm{div} b\|_{-2-\alpha} \|u\|_{2+\alpha}$$

and

$$\begin{aligned} \|b \odot \nabla u\|_{\alpha} &\lesssim \|b \prec u + b \circ u\|_{1+\alpha} + \|b\|_{\alpha} \|\nabla u\|_{\mathbb{L}^{\infty}} \\ &\lesssim \|b\|_{\alpha} (\|u\|_{1} + \|\nabla u\|_{\mathbb{L}^{\infty}_{T}}) \lesssim \|b\|_{\alpha} \|u\|_{2+\alpha}. \end{aligned}$$

Sketch of the proof

- Consider the following BKE:

$$\partial_t u + \Delta u + b \cdot \nabla u + f = 0, \quad u(T) = 0, \quad t \in [0, T].$$
$$b \in \mathbf{C}^{\alpha}, \quad u \in \mathbf{C}^{2+\alpha}.$$

- We define $b \cdot \nabla u := b \odot \nabla u + \mathrm{div} b \circ u + \mathrm{div} b \prec u$ where

$$\boxed{b \odot \nabla u := \mathrm{div}(b \prec u + b \circ u) + b \succ \nabla u.}$$

- The paraproduct implies that

$$\|\mathrm{div} b \circ u + \mathrm{div} b \prec u\|_{\alpha} \lesssim \|\mathrm{div} b\|_{-2-\alpha} \|u\|_{2+\alpha}$$

and

$$\begin{aligned}\|b \odot \nabla u\|_{\alpha} &\lesssim \|b \prec u + b \circ u\|_{1+\alpha} + \|b\|_{\alpha} \|\nabla u\|_{\mathbb{L}^{\infty}} \\ &\lesssim \|b\|_{\alpha} (\|u\|_{1} + \|\nabla u\|_{\mathbb{L}^{\infty}_{T}}) \lesssim \|b\|_{\alpha} \|u\|_{2+\alpha}.\end{aligned}$$

- Therefore, we have $u \in \mathbf{C}^{2+\alpha}$ and

$$\lim_{\delta \to 0} \sup_{|t-s| \le \delta, t, s \in [0,T]} \|\nabla u(t) - \nabla u(s)\|_{L^{\infty}} = 0.$$

- Zvonkin's transformation: taking $f = b$ and $\Phi_t(x) := x + u(t, x)$.

Weak solutions to supercritical SDEs with $\alpha = -1$

The setting

- We assume $d \geq 2$, $b \in L_T^q \mathbf{H}_p^{-1}$ with $p, q \in [2, \infty]$,

$$\frac{d}{p} + \frac{2}{q} < 1, \quad \mathrm{div} b = 0.$$

The setting

- We assume $d \geq 2$, $b \in L_T^q \mathbf{H}_p^{-1}$ with $p, q \in [2, \infty]$,

$$\frac{d}{p} + \frac{2}{q} < 1, \quad \mathrm{div} b = 0.$$

- Let $b_n \in \mathbf{C}_b^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ with $\lim_{n\to\infty} \|b_n - b\|_{L_T^q \mathbf{H}_p^{-1}} = 0$ and consider the following approximating SDE

$$X_t^n = X_0 + \int_0^t b_n(s, X_s^n)\mathrm{d}s + \sqrt{2}W_t.$$

- We denote the distribution of $(X_t^n)_{t\in[0,T]}$ by $\mathbb{P}_n \in \mathscr{P}(C([0,T]; \mathbb{R}^d))$.

Main results

Theorem 2 (H.-Zhang 2023)

i) For any $\mathscr{F}_0$ *measurable random variable* X_0, $\{\mathbb{P}_n\}_{n=1}^{\infty}$ *is* **tight** *in* $\mathscr{P}(C([0,T];\mathbb{R}^d))$.

ii) Moreover, if the distribution of X_0 *has an* L^2 *density w.r.t. the Lebesgue measure, then there is a continuous process* $(X_t)_{t\in[0,T]}$ *such that*

$$X_t = X_0 + \lim_{n\to\infty}\int_0^t b_n(r, X_r)\mathrm{d}r + \sqrt{2}W_t,$$

where the limit here is taken in $L^2(\Omega)$.

iii) Let $\mathbb{P}$ *be the law of the solution* $(X_t)_{t\in[0,T]}$. *The following almost surely Markov property holds: there is a Lebesgue zero set* $\mathscr{N}\subset(0,T)$ *such that for all* $s\in[0,T)\backslash\mathscr{N}$

$$\mathbb{E}_{\mathbb{P}}[f(\omega_t)|\mathscr{B}_s] = \mathbb{E}_{\mathbb{P}}[f(\omega_t)|\omega_s], \quad 0\le s\le t\le T,\ f\in\mathbf{C}_b(\mathbb{R}^d).$$

iv)When $b\in L^2([0,T]\times\mathbb{R}^d)$ *or* $b\in L_T^\infty\mathbf{B}_{\infty,2}^{-1}$ *(critical & ill-defined), there is only one accumulation point of* $\{\mathbb{P}_n\}_{n=1}^{\infty}$. *That is for any* $b_n\to b$, $\mathbb{P}_n$ *converges to the distribution of* $(X_t)_{t\in[0,T]}$.

Example: Particle system with singular kernels

- Consider the following singular interaction particle system in $\mathbb{R}^{Nd}$:

$$\mathrm{d}X_t^{N,i} = \sum_{j\neq i} \gamma_j K(X_t^{N,i} - X_t^{N,j})\mathrm{d}t + \sqrt{2}\mathrm{d}W_t^{N,i}, \quad i = 1, \cdots, N, \tag{2}$$

  where $K \in \mathbf{H}_\infty^{-1}(\mathbb{R}^d; \mathbb{R}^d)$ is divergence free, $W_t^{N,i}, i = 1, \cdots, N$ are N-independent standard d-dimensional Brownian motions, $\gamma_j \in \mathbb{R}$ and initial value has an L^2-density.
- (Jabin-Wang 2018) Existence of the related FPE and propagation of chaos. (The existence of a solution to the SDE (2) appears to be open).
- As a result, we have the weak existence to the N-particle system SDE (2).

Example: GFF and super-diffusive

- Let $d = 2$, $\varepsilon \in (0,1]$ and b_ε be a Gaussian field with

$$\mathbb{E} b_\varepsilon(f) b_\varepsilon(g) = \int_{|\xi| \leq 1/\varepsilon} \hat{f}(\xi)\hat{g}(-\xi)\Big(\mathbb{I}_{d\times d} - \frac{\xi \otimes \xi}{|\xi|^2}\Big)\mathrm{d}\xi.$$

- When $\varepsilon \to 0$, $b := \lim_\varepsilon b_\varepsilon$ formally satisfies

$$b := \nabla^\perp \xi := (-\partial_{x_2}\xi_1, \partial_{x_1}\xi_2) \in \mathbf{C}^{-1-} \quad \mathrm{div} b = 0,$$

  where $\xi = \xi(x)$ is the two-dimensional Gaussian Free Field (GFF)
- (Super-diffusive)
  When $\varepsilon = 1$, $\mathbb{E}|X_t|^2 \asymp t\sqrt{\ln t}$
  (Cannizzaro-HaunschmidSibitz-Toninelli 2022)
  (Chatzigeorgiou-Morfe-Otto-Wang 2022).
- For any $p \in (2, \infty)$

$$\sup_{\varepsilon < 1/2} \Big\| \frac{b_\varepsilon}{\sqrt{\ln \varepsilon}} \Big\|_{\mathbf{H}^{-1}_{p,loc}} < \infty, \quad a.s.$$

  By our results, one sees that the solutions $\{X^\varepsilon_t\}_{[0,T]}$ to the following approximation SDEs is tight

$$\mathrm{d}X^\varepsilon_t = \frac{b_\varepsilon(X^\varepsilon_t)}{\sqrt{\ln \varepsilon}}\mathrm{d}t + \sqrt{2}\mathrm{d}W_t.$$

Sketch of the proof– Tightness

- Consider the following backward PDE

$$\partial_t u + \Delta u + b \cdot \nabla u + f = 0, \quad u(T) = 0 \tag{PDE}$$

and the following approximation PDEs

$$\partial_t u_n + \Delta u_n + b_n \cdot \nabla u_n + f = 0, \quad u(T) = 0 \tag{APDE}$$

- Under the condition **($\mathbf{H}^{sup}$)**, by De Giorgi's method in (Zhang-Zhao 2021), we have

$$\sup_n \left(\|u_n\|_\infty + \|\nabla u_n\|_2\right) < \infty,$$

which implies the there is a weak solution u to (PDE).

- (Problem): Since we don't know whether $\langle u, b \cdot \nabla u \rangle = 0$ holds a priority, we don't have the uniqueness of (PDE).

Sketch of the proof– Tightness

- Consider the following backward PDE

$$\partial_t u + \Delta u + b \cdot \nabla u + f = 0, \quad u(T) = 0 \tag{PDE}$$

and the following approximation PDEs

$$\partial_t u_n + \Delta u_n + b_n \cdot \nabla u_n + f = 0, \quad u(T) = 0 \tag{APDE}$$

- Under the condition $(\mathbf{H}^{sup})$, by De Giorgi's method in (Zhang-Zhao 2021), we have

$$\sup_n (\|u_n\|_\infty + \|\nabla u_n\|_2) < \infty,$$

which implies the there is a weak solution u to (PDE).
- (Problem): Since we don't know whether $\langle u, b \cdot \nabla u \rangle = 0$ holds a priority, we don't have the uniqueness of (PDE).
- By Itô's formula,

$$\sup_n \left| \mathbb{E} \int_0^T f(r, X_r^n) \mathrm{d}r \right| \leq \|u_n\|_\infty \lesssim \|f\|_{L_T^q \mathbf{H}_p^{-1}} \quad \text{(1st Krylov estimate).}$$

Sketch of the proof– Tightness

- By Aldous' criterion of tightness and the strong Markov property, we only need to show

$$\lim_{\delta\to 0}\sup_{x_0\in\mathbb{R}^d}\sup_{\tau\leqslant\delta}\sup_{n}\mathbf{E}|X^n_\tau(x_0)-x_0|=0.$$

Sketch of the proof– Tightness

- By Aldous' criterion of tightness and the strong Markov property, we only need to show

$$\lim_{\delta\to 0}\sup_{x_0\in\mathbb{R}^d}\sup_{\tau\leqslant\delta}\sup_n \mathbf{E}|X^n_\tau(x_0)-x_0|=0.$$

- Fix $\varepsilon\in(0,1)$. Define

$$h_\varepsilon(x):=\sqrt{\varepsilon^2+|x-x_0|^2},\ \ |\nabla h_\varepsilon|\leqslant C,\ \ |\nabla^2 h_\varepsilon|\leqslant C\varepsilon^{-1}.$$

Sketch of the proof– Tightness

- By Aldous' criterion of tightness and the strong Markov property, we only need to show
$$\lim_{\delta\to 0}\sup_{x_0\in\mathbb{R}^d}\sup_{\tau\leqslant\delta}\sup_n \mathbf{E}|X^n_\tau(x_0)-x_0|=0.$$
- Fix $\varepsilon\in(0,1)$. Define
$$h_\varepsilon(x):=\sqrt{\varepsilon^2+|x-x_0|^2},\ \ |\nabla h_\varepsilon|\leqslant C,\ \ |\nabla^2 h_\varepsilon|\leqslant C\varepsilon^{-1}.$$
- By Itô's formula, we have
$$\begin{aligned}\mathbf{E}|X^n_\tau-x_0| &\leqslant \mathbf{E}h_\varepsilon(X^n_\tau)=\varepsilon+\mathbf{E}\left(\int_0^\tau(\Delta+b_n(s)\cdot\nabla)h_\varepsilon(X^n_s)\mathrm{d}s\right)\\ &\lesssim \varepsilon+\delta\varepsilon^{-1}+\left|\mathbf{E}\left(\int_0^\tau(b_n\cdot\nabla h_\varepsilon)(s,X^n_s)\mathrm{d}s\right)\right|\\ &\overset{1stKE}{\lesssim} \varepsilon+\delta\varepsilon^{-1}+\|b_n\cdot\nabla h_\varepsilon\|_{\mathbb{L}^q_\delta\mathbf{H}^{-1}_p}\left(\lesssim \|b_n\|_{\mathbb{L}^q_\delta\mathbf{H}^{-1}_p}\|\nabla h_\varepsilon\|_{\mathbf{C}^1_b}\right)\\ &\lesssim \varepsilon+\delta\varepsilon^{-1}+\|b\|_{\mathbb{L}^q_\delta\mathbf{H}^{-1}_p}\quad\to 0\end{aligned}$$
as $\delta\to 0$ and $\varepsilon\to 0$.

Sketch of the proof– Weak existence

- Tightness + Skorokhod's representation theorem $\Rightarrow$ limit process $(X_t)_{t\in[0,T]}$.
- What we need : $\lim_{n\to\infty}\sup_{m\geq n}\mathbb{E}|\int_0^t(b_n-b_m)(s,X_s)\mathrm{d}s|=0$.

Sketch of the proof– Weak existence

- Tightness + Skorokhod's representation theorem $\Rightarrow$ limit process $(X_t)_{t\in[0,T]}$.
- What we need : $\lim_{n\to\infty}\sup_{m\geq n}\mathbb{E}|\int_0^t(b_n-b_m)(s,X_s)\mathrm{d}s|=0$.
- The second Krylov type estimate:

$$\sup_n I_n(f):=\sup_n \mathbb{E}\left|\int_0^t f(s,X_s^n)\mathrm{d}s\right|^2\lesssim \|f\|^2_{L_T^q\mathbf{H}_p^{-1}}.$$

Sketch of the proof– Weak existence

- Tightness + Skorokhod's representation theorem $\Rightarrow$ limit process $(X_t)_{t\in[0,T]}$.
- What we need : $\lim_{n\to\infty}\sup_{m\geq n}\mathbb{E}|\int_0^t(b_n-b_m)(s,X_s)\mathrm{d}s|=0$.
- The second Krylov type estimate:

$$\sup_n I_n(f) := \sup_n \mathbb{E}\left|\int_0^t f(s,X_s^n)\mathrm{d}s\right|^2 \lesssim \|f\|^2_{L_T^q\mathbf{H}_p^{-1}}.$$

  - Recall the following approximation BKE

$$\partial_s u_n + \Delta u_n + b_n\cdot\nabla u_n + f = 0, \quad u(t)=0$$

  and consider the following FPE

$$\partial_s\rho_n = \Delta\rho_n - \mathrm{div}(b_n\rho_n).$$

Sketch of the proof– Weak existence

- Tightness + Skorokhod's representation theorem $\Rightarrow$ limit process $(X_t)_{t\in[0,T]}$.
- What we need : $\lim_{n\to\infty}\sup_{m\geq n}\mathbb{E}|\int_0^t(b_n-b_m)(s,X_s)\mathrm{d}s|=0$.
- The second Krylov type estimate:

$$\sup_n I_n(f) := \sup_n \mathbb{E}\left|\int_0^t f(s,X_s^n)\mathrm{d}s\right|^2 \lesssim \|f\|^2_{L_T^q\mathbf{H}_p^{-1}}.$$

  - Recall the following approximation BKE

$$\partial_s u_n + \Delta u_n + b_n\cdot\nabla u_n + f = 0, \quad u(t) = 0$$

    and consider the following FPE

$$\partial_s\rho_n = \Delta\rho_n - \mathrm{div}(b_n\rho_n).$$

  - By the representation of the solution to BKE,

$$\begin{aligned} I_n(f) &= 2\mathbb{E}\int_0^t\int_s^t f(s,X_s^n)f(r,X_r^n)\mathrm{d}r\mathrm{d}s \\ &= 2\mathbb{E}\int_0^t f(s,X_s^n)\mathbb{E}^{\mathscr{F}_s}\left[\int_s^t f(r,X_r^n)\mathrm{d}r\right]\mathrm{d}s \\ &= 2\mathbb{E}\int_0^t f(s,X_s^n)u_n(s,X_s^n)\mathrm{d}s = 2\int_0^t\langle f(s)u_n(s),\rho_n(s)\rangle\mathrm{d}s \\ &\lesssim \|f\|_{L_T^q\mathbf{H}_p^{-1}}\|u_n\|_{L_t^\infty L^2\cap L_T^2\mathbf{H}_2^1}\|\rho_n\|_{L_T^\infty L^2\cap L_T^2\mathbf{H}_2^1} \lesssim \|f\|^2_{L_T^q\mathbf{H}_p^{-1}}\|\rho_0\|_2. \end{aligned}$$

Sketch of the proof–Markov property

- Idea: obtain the uniqueness martingale solution.

Definition 3 (Martingale solution)

Let $\mu \in \mathscr{P}(\mathbb{R}^d)$. We call a probability measure $\mathbb{P} \in \mathscr{P}(\mathbb{C}_T)$ a martingale solution of SDE (1) starting from μ, if $\mathbb{P} \circ (\omega_0)^{-1} = \mu$ and for any $f \in C_c^\infty([0,T] \times \mathbb{R}^d)$,

$$M_t^f := u(t, \omega_t) - u(0, \omega_0) - \int_0^t f(r, \omega_r) \mathrm{d}r, \;\; \omega_\cdot \in \mathbb{C}_T,$$

is a martingale under $\mathbb{P}$ with respect to the natural filtration $\mathscr{B}_s$.

Sketch of the proof–Markov property

- Idea: obtain the uniqueness martingale solution.

Definition 3 (Martingale solution)

Let $\mu \in \mathscr{P}(\mathbb{R}^d)$. We call a probability measure $\mathbb{P} \in \mathscr{P}(\mathbb{C}_T)$ a martingale solution of SDE (1) starting from μ, if $\mathbb{P} \circ (\omega_0)^{-1} = \mu$ and for any $f \in C_c^\infty([0,T] \times \mathbb{R}^d)$,

$$M_t^f := u(t, \omega_t) - u(0, \omega_0) - \int_0^t f(r, \omega_r)\mathrm{d}r, \ \ \omega_\cdot \in \mathbb{C}_T,$$

is a martingale under $\mathbb{P}$ with respect to the natural filtration $\mathscr{B}_s$.

- Problem: There is **no** uniqueness to (PDE).
- We couldn't show the existence of a solution to the martingale solution such that the definition holds for all solutions u.

Sketch of the proof–Markov property

- Idea: obtain the uniqueness martingale solution.

Definition 3 (Martingale solution)

Let $\mu \in \mathscr{P}(\mathbb{R}^d)$. We call a probability measure $\mathbb{P} \in \mathscr{P}(\mathbb{C}_T)$ a martingale solution of SDE (1) starting from μ, if $\mathbb{P} \circ (\omega_0)^{-1} = \mu$ and for any $f \in C_c^\infty([0,T] \times \mathbb{R}^d)$,

$$M_t^f := u(t, \omega_t) - u(0, \omega_0) - \int_0^t f(r, \omega_r)\mathrm{d}r, \quad \omega_\cdot \in \mathbb{C}_T,$$

is a martingale under $\mathbb{P}$ with respect to the natural filtration $\mathscr{B}_s$.

- Problem: There is **no** uniqueness to (PDE).
- We couldn't show the existence of a solution to the martingale solution such that the definition holds for all solutions u.
- We can find a bounded linear operator

$$\boxed{\mathcal{S} : L_T^q \mathbf{H}_p^{-1} \to L_T^\infty L^\infty \cap L_T^2 \mathbf{H}_2^1}$$

  such that for any f, $u = \mathcal{S}f$ solves (PDE).

  - Once $b \in L_T^\infty \mathbf{B}_{\infty,2}^{-1}$, we have the uniqueness and stability for (PDE), which implies the uniqueness of the operator $\mathcal{S}$.

Sketch of the proof–Markov property

Definition 4 (Generalized martingale solution)

Let $\mu \in \mathscr{P}(\mathbb{R}^d)$. We call a probability measure $\mathbb{P} \in \mathscr{P}(\mathbb{C}_T)$ a generalized martingale solution of SDE (1) starting from μ and associated with the operator $\mathcal{S}$, if $\mathbb{P} \circ (\omega_0)^{-1} = \mu$ and for any $f \in C_c^\infty([0,T] \times \mathbb{R}^d)$,

$$M_t^f := \mathcal{S}_f(t, \omega_t) - \mathcal{S}_f(0, \omega_0) - \int_0^t f(r, \omega_r)\mathrm{d}r, \quad w_\cdot \in \mathbb{C}_T,$$

is a martingale under $\mathbb{P}$ with respect to the natural filtration $\mathscr{B}_s$.

Sketch of the proof–Markov property

Definition 4 (Generalized martingale solution)

Let $\mu \in \mathscr{P}(\mathbb{R}^d)$. We call a probability measure $\mathbb{P} \in \mathscr{P}(\mathbb{C}_T)$ a generalized martingale solution of SDE (1) starting from μ and associated with the operator $\mathcal{S}$, if $\mathbb{P} \circ (\omega_0)^{-1} = \mu$ and for any $f \in C_c^\infty([0,T] \times \mathbb{R}^d)$,

$$M_t^f := \mathcal{S}_f(t, \omega_t) - \mathcal{S}_f(0, \omega_0) - \int_0^t f(r, \omega_r)\mathrm{d}r, \quad w_\cdot \in \mathbb{C}_T,$$

is a martingale under $\mathbb{P}$ with respect to the natural filtration $\mathscr{B}_s$.

Theorem 3 (H.-Zhang 2023)

Assume μ has an L^2 density w.r.t. the Lebesgue measure. There is a unique generalized martingale solution w.r.t. the $\mathcal{S}$.

- We can find a subsequence $\{n_k\}_{k=1}^\infty$ such that $u_{n_k} \to \mathcal{S}f$ ($\mathcal{S}$ depends on this subsequence). Then the law of a weak solution is just a generalized martingale solution. The Markov property follows from the definition of the generalized martingale solution.

Further works

- Uniqueness in the supercritical cases (Counterexample by Zhao (2019)).
- Characterize the limit of the approximation solutions to the SDEs with drift $b = \nabla^{\perp}\mathrm{GFF}$.
- $\cdots$

Thank you!