Gradient estimate for SDEs driven by cylindrical Lévy processes

Zimo Hao[1]

Based on a joint work with Zhen-Qing Chen[2,3] and Xicheng Zhang[1]

[1]Wuhan University
[2]University of Washington
[3]Beijing Institute of Technology

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Plan of the talk

1. **Introduction**
   - Introduction of cylindrical α-stable process
   - SDE driven by cylindrical α-stable process
   - Well-known results
2. **Main Results**
   - Introduction of Besov space
3. **Sketch of the proof**
   - Crucial lemma
   - Hölder estimate
   - Lift and gradient estimate
4. **Future works**

Part 1 : Introduction and our questions

α-stable processes

- Fix $\alpha \in (0,2)$. Let L_t^α be an α-stable process with Lévy measure and ν_α with the following form:

$$\nu_\alpha(\mathrm{d}y) := \frac{\mathrm{d}r\mu(\mathrm{d}\omega)}{r^{1+\alpha}}, \ \ y = r\omega,$$

where μ is a finite measure on the unit sphere $\mathbb{S}^{d-1}$.

- L_t^α has the following scaling property:

$$(\lambda^{1/\alpha} L_{\lambda t})_{t\geqslant 0} \stackrel{(d)}{=} (L_t)_{t\geqslant 0}.$$

- We call L_t^α being non-degenerate if the Lévy measure ν_α is nondegenerate in the following sense:

$$\int_{\mathbb{S}^{d-1}} |\ \omega_0 \cdot \omega\ |\ \mu(\mathrm{d}\omega) \neq 0, \ \ \forall \omega_0 \in \mathbb{S}^{d-1}. \tag{1.1}$$

α-stable processes

- It is easy to verify that for any $\beta_1 < \alpha < \beta_2$, positive number λ and positive measurable function f,

$$\int_{\mathbb{R}^d} \left(|y|^{\beta_1} \wedge |y|^{\beta_2} \right) \nu_\alpha(\mathrm{d}y) < \infty,$$

$$\int_{\mathbb{R}^d} f(\lambda y) \nu_\alpha(\mathrm{d}y) = \lambda^\alpha \int_{\mathbb{R}^d} f(y) \nu_\alpha(\mathrm{d}y).$$

- By Lévy-Khintchine formula, we have

$$\mathbb{E}(\exp[i L_t^\alpha \xi]) = e^{t\psi(\xi)}$$

where

$$\psi(\xi) := \int_{\mathbb{R}^d} (e^{i\xi \cdot z} - 1 - i\xi \cdot z \mathbb{1}_{|z|<1}) \nu_\alpha(dz).$$

Example 1

When μ is Lebesgue measure of $\mathbb{S}^{d-1}$,

$$\nu_\alpha(dz) = |z|^{-d-\alpha}dz, \quad \psi(\xi) = -c|\xi|^\alpha \in C^\infty(\mathbb{R}^d \backslash \{0\}).$$

We call this process the standard d-dim α-stable process.

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Example 2

When

$$\mu = \sum_{i=1}^{d} \delta_{e_i},$$

where $\{e_i\}_{i=1}^d$ is a basis of $\mathbb{R}^d$ with $e_i = (0, ..1(i\text{th}), .., 0)$. It is the Lévy measure of process $(L_t^1, L_t^2, ..., L_t^d)$, where $\{L_t^i\}_{i=1}^d$ are i.i.d. standard 1-dim α-stable processes.

$$\psi(\xi) = -c\sum_{i=1}^{d} |\xi_i|^\alpha \in C^\infty(\mathbb{R}^d \backslash \cup_{i=1}^d \mathbb{R}_i^{d-1}),$$

where $\mathbb{R}_i^{d-1} := \{\xi_i = 0\}$.
We call this process the cylindrical α-stable process.

SDE driven by cylindrical α-stable process

Consider the following SDE driven by the Lévy process L^{ν_α},

$$\begin{cases} \mathrm{d}X_t^x = \int_{\mathbb{R}^d} \sigma(X_{t-}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z) + b(X_t)\mathrm{d}t, \\ X_0^x = x, \end{cases} \tag{1.2}$$

where $\sigma = (\sigma_i)_{i=1}^d : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $b = (b_k)_{k=1}^d : \mathbb{R}^d \to \mathbb{R}^d$, $N(\mathrm{d}t, \mathrm{d}z)$ is the Poisson random measure of L_t^α and

$$\tilde{N}(\mathrm{d}t, \mathrm{d}z) := N(\mathrm{d}t, \mathrm{d}z) - \mathrm{d}t\nu_\alpha(\mathrm{d}z).$$

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Example 3

When L_t^α is a cylindrical α-stable process with $\alpha < 1$, the infinitesimal generator of X_t^x is $\mathscr{L}_c^\alpha$,

$$\mathscr{L}_c^\alpha f(x) = \sum_{i=1}^d \int_{\mathbb{R}} \frac{f(x + \sigma_i(x, z)) - f(x)}{|z|^{1+\alpha}} \mathrm{d}z.$$

Questions

- In what condition of σ and b, there is a weak(or strong) solution of SDE (1.2)?
- If there is a weak solution, does the solution have (strong) Feller property?
- If the solution has (strong) Feller property, can we get the precise estimate?

Assumptions

($\mathbf{H}_s^\sigma$) $\sigma(x,z) = A(x)z$ for some matrix value map $A = (a_{i,j})\ \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, there is a positive number c_0 such that for any $x, y, \xi \in \mathbb{R}^d$ and all $i, j = 1, ..., d$

$$c_0^{-1}|\xi| \leqslant |\xi \cdot A(x)\xi| \leqslant c_0|\xi|, \tag{1.3}$$

$$|a_{i,j}(x) - a_{i,j}(y)| \leqslant c_0|x - y|. \tag{1.4}$$

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($\mathbf{H}_\beta^b$) For $\beta \in (0,1)$,

$$\sup_{0<|x-y|\leqslant 1} \frac{|b(x)-b(y)|}{|x-y|^\beta} < \infty. \tag{1.5}$$

- Notice that condition ($\mathbf{H}_{\beta_1}^b$) implies ($\mathbf{H}_{\beta_2}^b$) if $\beta_1 \geqslant \beta_2$ and $b(x) = x$ satisfies (1.5) for all $\beta \in (0,1)$.
- We always assume that there is a weak solution X_t^x of SDE (1.2) and define

$$P_t^{\sigma,b}\psi(x) = \mathbb{E}(\psi(X_t^x)), \qquad P_t^\sigma := P_t^{\sigma,0}.$$

Well-known results

2006 (Bass-Chen)

There is a weak solution X_t^x of (1.2) when σ is continuous, $b \equiv 0$ and L_t^ν is a cylindrical α-stable process.

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Assume $\sigma(x,z) = \sigma(x)z$ is continuous in variable x, $b \equiv 0$ and L_t^ν is a cylindrical α-stable process. If any bounded function h satisfies

$$h(x) = \mathbb{E}[h(X_{\tau_D}^x)] \quad \text{for every } x \in D$$

for some bounded domain D, then h is Hölder continuous in D.

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2017 (Chen-Zhang-Zhao)
Under the condition $(\mathbf{H}_s^\sigma)$ and $(\mathbf{H}_\beta^b)$ with $\beta \in (1 - \frac{\alpha}{2}, 1)$, there is a unique strong solution of (1.2).

Well-known results

2018 (Kulczycki-Ryznar-Sztonyk)

Assume $b \equiv 0$ and L_t^ν is a cylindrical α-stable process with $\alpha \in (0,1)$. Under the condition **($\mathbf{H}_s^\sigma$)**, for any $\gamma \in (0,\alpha)$, $T > 0$, there is a constant C such that for all $t \in (0,T]$, $x, y \in \mathbb{R}^d$ and $f \in L^\infty(\mathbb{R}^d)$

$$|P_t^\sigma f(x) - P_t^\sigma f(y)| \leqslant C|x-y|^\gamma t^{-\frac{\gamma}{\alpha}} \|f\|_{L^\infty}. \tag{1.6}$$

For any $\gamma \in (0, \frac{\alpha}{d})$, $T > 0$, there is a constant C such that for all $t \in (0,T]$, $x \in \mathbb{R}^d$ and $f \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$

$$|P_t^\sigma f(x)| \leqslant C t^{-\frac{\gamma d}{\alpha}} \|f\|_{L^\infty}^{1-\gamma} \|f\|_{L^1}^{\gamma}. \tag{1.7}$$

Part 2: Our main results

Littlewood-Paley decomposition and Besov space

- Let ϕ_0 be a radial C^∞-function on $\mathbb{R}^d$ with
$$\phi_0(\xi) = 1 \ \text{ for } \ \xi \in B_1 \ \text{ and } \ \phi_0(\xi) = 0 \ \text{ for } \ \xi \notin B_2.$$
- For $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^d$ and $j \in \mathbb{N}$, define
$$\phi_j(\xi) := \phi_0(2^{-j}\xi) - \phi_0(2^{-(j-1)}\xi).$$
- It is easy to see that for $j \in \mathbb{N}$, $\phi_j(\xi) = \phi_1(2^{-(j-1)}\xi) \geqslant 0$ and
$$\text{supp}\phi_j \subset B_{2^{j+1}} \setminus B_{2^{j-1}}, \ \sum_{j=0}^{k} \phi_j(\xi) = \phi_0(2^{-k}\xi) \to 1, \ \ k \to \infty.$$

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Definition 4 (Besov spaces)

For given $j \in \mathbb{N}_0$, the block operator Δ_j is defined on $\mathscr{S}'$ by
$$\Delta_j f(x) := (\phi_j \hat{f})^{\vee}(x) = \check{\phi}_j * f(x) = 2^{\cdot m(j-1)} \int_{\mathbb{R}^d} \check{\phi}_1(2^{(j-1)}y) f(x-y) \mathrm{d}y.$$
For any $s \in \mathbb{R}$ and $p \in [1, \infty]$, the Besov space $B^s_{p,\infty}$ is defined by
$$B^s_{p,\infty}(\mathbb{R}^d) := \left\{ f \in \mathscr{S}'(\mathbb{R}^d) : \|f\|_{B^s_{p,\infty}} := \sup_{j \geqslant 0} \left(2^{sj} \|\Delta_j f\|_{L^p} \right) < \infty \right\}.$$

The Propositions of Besov Space

Proposition 5

For any $s_1 \geqslant 0$ *and* $s_2 > 0$ *with* $s_2 \notin \mathbb{N}$,

$$H^{s_1,p}(\mathbb{R}^d) \subset B^{s_1}_{p,\infty}(\mathbb{R}^d) \quad \textit{and} \quad C^{s_2}(\mathbb{R}^d) = B^{s_2}_{\infty,\infty}(\mathbb{R}^d),$$

For any $n \in \mathbb{N}$,

$$C^n(\mathbb{R}^d) \subset B^n_{\infty,\infty}(\mathbb{R}^d).$$

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For any $n \in \mathbb{N}$,

$$C^n(\mathbb{R}^d) \subset B^n_{\infty,\infty}(\mathbb{R}^d).$$

- For $j \in \mathbb{N}_0$, by definition it is easy to see that

$$\Delta_j = \Delta_j \widetilde{\Delta}_j, \ \text{ where } \ \widetilde{\Delta}_j := \Delta_{j-1} + \Delta_j + \Delta_{j+1} \text{ with } \Delta_{-1} \equiv 0, \qquad (2.1)$$

and Δ_j is symmetric in the sense that

$$\langle \Delta_j f, g \rangle = \langle f, \Delta_j g \rangle.$$

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- The cut-off low frequency operator S_k is defined by

$$S_k f := \sum_{j=0}^{k-1} \Delta_j f = 2^{dk} \int_{\mathbb{R}^d} \check{\phi}_0(2^k(x-y)) f(y) \mathrm{d}y \to f. \tag{2.2}$$

Our assumption for σ

($\mathbf{H}_\mu^\sigma$) There is a constant $c_0 > 1$ such that for all $x, y, z \in \mathbb{R}^d$ and all $\lambda > 0$

$$\inf_{\omega \in \mathbb{S}^{d-1}} \inf_{\lambda > 0} \lambda \int_{\mathbb{S}^{d-1}} |\omega \cdot \sigma(x, \frac{z}{\lambda})| \mu(\mathrm{d}z) \geqslant c_0^{-1}, \tag{2.3}$$

$$|\sigma(x, z) - \sigma(y, z)| \leqslant c_0 |x - y||z|.$$

$$c_0^{-1}|z| \leqslant |\sigma(x, z)| \leqslant c_0 |z|.$$

Remark 6

- *Notice that condition $\boldsymbol{H}_s^\sigma$ implies condition $\boldsymbol{H}_\mu^\sigma$ here.*
- $\sigma(x, z) = (2 + sin z_1)z$ *satisfies condition $\boldsymbol{H}_\mu^\sigma$ but not satisfies condition $\boldsymbol{H}_s^\sigma$.*

Main Results

Theorem 7 (Gradient estimate for SDE solution)

Assume one of the following conditions holds,

- $\alpha \in (0,2)$, $b \equiv 0$ *and let* $\beta = 1$.
- $\alpha \in (\frac{1}{2}, 2)$ *and condition* $(\mathbf{H}^b_\beta)$ *holds with* $\beta \in ((1-\alpha) \vee 0, \alpha)$.

Under condition $(\mathbf{H}^\sigma_\mu)$, *for any*

$$\gamma_1 \in [0, \alpha + \alpha \wedge \beta), \quad (\tfrac{d}{p} - \gamma_2) \in \Big(-\gamma_1, \alpha - (\alpha - 1) \wedge (1 - \beta) \Big),$$

there is a constant C_T *such that for all* $\phi \in C_0^\infty(\mathbb{R}^d)$ *and all* $t \in (0, T]$,

$$\|P^{\sigma,b}_t \phi\|_{B^{\gamma_1}_{\infty,\infty}} \leqslant C_T t^{-\frac{1}{\alpha}\left(\frac{d}{p} - \gamma_2 + \gamma_1\right)} \|\phi\|_{B^{\gamma_2}_{p,\infty}}. \tag{2.4}$$

In particular, when $\alpha > \frac{1}{2}$, *we obtain the gradient estimate,*

$$\|\nabla P^{\sigma,b}_t \phi\|_\infty \leqslant C_T t^{-\frac{1}{\alpha}} \|\phi\|_\infty.$$

Main Results

- If taking $\gamma_2 = 0$ and $p_1 = \infty$, we get a corollary directly.

Corollary 8

Assume one of the following conditions holds,

- $\alpha \in (0,2)$, $b \equiv 0$ *and let* $\beta = 1$.
- $\alpha \in (\frac{1}{2}, 2)$ *and condition* $(\mathbf{H}^b_\beta)$ *holds with* $\beta \in ((1-\alpha) \vee 0, \alpha)$.

Under condition $(\mathbf{H}^\sigma_\mu)$, *for any* $\gamma \in [0, \alpha + \alpha \wedge \beta)$, *there is a constant* C_T *such that for all* $\phi \in L^\infty(\mathbb{R}^d)$ *and all* $t \in (0, T]$,

$$\|P_t^{\sigma,b}\phi\|_{C^\gamma} \leqslant C_T t^{-\frac{\gamma}{\alpha}} \|\phi\|_{L^\infty}. \tag{2.5}$$

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$$\|P^{\sigma,b}_t \phi\|_{C^\gamma} \leqslant C_T t^{-\frac{\gamma}{\alpha}} \|\phi\|_{L^\infty}. \tag{2.5}$$

- Notice that (2.5) reduced the restriction of the γ in (1.6) from $(0, \alpha)$ to $(0, \alpha+\alpha\wedge\beta)$. In particular, we have gradient estimate. Moreover, we can deal with the case $\alpha \geqslant 1$.
- By a way of interpolation, we also get (1.7) from Theorem7.

Main Results

- By the dual space theorem, we get the existence of density:

Corollary 9

Fix any $0 \leqslant s < t$, *there is a function* $p^{\sigma,b}_{s,t}\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, *for any* $x \in \mathbb{R}^d$

$$p^{\sigma,b}_{s,t}(x,\cdot) \in \cap_{(s,r)\in\mathscr{I}} H^{s,q}(\mathbb{R}^d)$$

where $H^{s,q}$ *is the Sobolev space and*

$$\mathscr{I} = \left\{(s,r) \mid s < \alpha - (\alpha-1)\vee(1-\beta), \quad q \in [1, \tfrac{d}{d-\alpha+s+(\alpha-1)\vee(\beta-1)})\right\}$$
$$\subset [0,\infty] \times [1,\infty],$$

such that all $\phi \in C^\infty_0(\mathbb{R}^d)$,

$$P^{\sigma,b}_t \phi(x) = \int_{\mathbb{R}^d} p^{\sigma,b}_t(x,y)\phi(y)\mathrm{d}y.$$

Crucial lemma

- Let $\theta : \mathbb{R}_+ \to \mathbb{R}^d$ is a measurable function and $p_{s,t}$ be the transition probability of process

$$Z_{s,t} := \int_s^t \int_{\mathbb{R}^d} \sigma(\theta(r), z)\tilde{N}(dz, dr).$$

Lemma 10 (Crucial Lemma)

- *For any $\beta \in [0, \alpha)$, $\gamma \in [0, +\infty)$ and $T > 0$, there is a constants C such that for $m \in \mathbb{N}_0$ all $j > 0$, $f \in L^1_{loc}(\mathbb{R}_+)$ and $t \in (0, T]$ $s \in [0, t)$,*

$$\int_0^t \int_{\mathbb{R}^d} |x|^\beta |\nabla^m \Delta_j p_{s,t}(x)||f(s)|\mathrm{d}x\mathrm{d}s \leqslant C 2^{(m-\gamma-\beta)j} \int_0^t (t-s)^{-\frac{\gamma}{\alpha}} |f(s)|\mathrm{d}s.$$

- *For any $m \in \mathbb{N}_0$, $q \in [1, \infty]$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\gamma \in [0, +\infty)$, there is a constant C such that for all $(t-s) \in (0, T]$,*

$$\|\nabla^m \Delta_j p_{s,t}\|_{L^q(\mathbb{R}^d)} \leqslant C(t-s)^{-\frac{1}{\alpha}(\gamma - m + \frac{d}{p})} 2^{-\gamma j}.$$

The key point of proof

- For simplify, we assume $\sigma(x, z) = A(x)z$ for some matrix value map $A : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $a(t) := A(\theta(t))$. Recall that $p_{s,t}$ is the transition probability of

$$Z_{s,t} = \int_s^t a(r)\mathrm{d}L_t^\alpha \quad \text{with} \quad \lambda^{\frac{1}{\alpha}} L_{\lambda t}^\alpha \overset{(d)}{=} L_t^\alpha.$$

Therefore using the change of variable and the scaling property, we have

$$\begin{aligned}\int_s^t a(r)\mathrm{d}L_r^\alpha &= \int_0^{t-s} a(r+s)\mathrm{d}\Big(L_{r+s}^\alpha - L_s^\alpha\Big) \\ &\overset{(d)}{=} (t-s)^{-\frac{1}{\alpha}} \int_0^1 a(r(t-s)+s)\mathrm{d}L_r^\alpha.\end{aligned}$$

We denote by $\bar{p}_{0,1}$ the density of $\int_0^1 a(r(t-s)+s)\mathrm{d}L_r^\alpha$, then

$$p_{s,t}(x) = (t-s)^{-\frac{d}{\alpha}} \bar{p}_{0,1}((t-s)^{-\frac{1}{\alpha}} x).$$

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$$Z_{s,t} = \int_s^t a(r)\mathrm{d}L_t^\alpha \quad \text{with} \quad \lambda^{\frac{1}{\alpha}} L^\alpha_{\lambda t} \overset{(d)}{=} L_t^\alpha.$$
Therefore using the change of variable and the scaling property, we have
$$\int_s^t a(r)\mathrm{d}L_r^\alpha = \int_0^{t-s} a(r+s)\mathrm{d}\Big(L^\alpha_{r+s} - L^\alpha_s\Big)$$
$$\overset{(d)}{=} (t-s)^{-\frac{1}{\alpha}} \int_0^1 a(r(t-s)+s)\mathrm{d}L_r^\alpha.$$
We denote by $\bar{p}_{0,1}$ the density of $\int_0^1 a(r(t-s)+s)\mathrm{d}L_r^\alpha$, then
$$p_{s,t}(x) = (t-s)^{-\frac{d}{\alpha}} \bar{p}_{0,1}((t-s)^{-\frac{1}{\alpha}} x).$$
- Condition
$$\inf_{\omega\in\mathbb{S}^{d-1}} \inf_{\lambda>0} \lambda \int_{\mathbb{S}^{d-1}} |\omega\cdot\sigma(x, \frac{z}{\lambda})|\mu(\mathrm{d}z) \geqslant c_0^{-1} \tag{3.1}$$
guarantee that for any $n \in \mathbb{N}_0$ and $\beta \in [0,\alpha)$, there is a constant C such that
$$\int_{\mathbb{R}^d} |x|^\beta |\nabla^n \bar{p}_{0,1}(x)|\mathrm{d}x \leqslant C.$$

PDE related to SDE

- Recall that X_t^x is the weak solution of SDE

$$\begin{cases} \mathrm{d}X_t^x = \int_{\mathbb{R}^d} \sigma(X_{t-}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z) + b(X_t)\mathrm{d}t, \\ X_0^x = x. \end{cases}$$

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- We let $u(t,x) \in C([0,T]; C^{\alpha+\varepsilon}(\mathbb{R}^d) \cap C^{1+\varepsilon}(\mathbb{R}^d))$ for some $\varepsilon > 0$ be a solution of following PDE,

$$\begin{cases} \partial_t u(t,x) = \mathscr{L}_\sigma^\alpha u(t,x) + b(x) \cdot \nabla u(t,x), \\ u(0,x) = \phi(x), \end{cases} \tag{3.2}$$

where

$$\mathscr{L}_\sigma^\alpha u(t,x) = \int_{\mathbb{R}^d} \Big(u(t, x + \sigma(x,z)) - u(t,x) - \mathbb{1}_{\alpha \geqslant 1} \sigma(x,z) \cdot \nabla u(t,x) \Big) \nu_\alpha(\mathrm{d}z).$$

PDE related to SDE

- Recall that X_t^x is the weak solution of SDE

$$\begin{cases} \mathrm{d}X_t^x = \int_{\mathbb{R}^d} \sigma(X_{t-}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z) + b(X_t)\mathrm{d}t, \\ X_0^x = x. \end{cases}$$

- We let $u(t,x) \in C([0,T]; C^{\alpha+\varepsilon}(\mathbb{R}^d) \cap C^{1+\varepsilon}(\mathbb{R}^d))$ for some $\varepsilon > 0$ be a solution of following PDE,

$$\begin{cases} \partial_t u(t,x) = \mathscr{L}_\sigma^\alpha u(t,x) + b(x) \cdot \nabla u(t,x), \\ u(0,x) = \phi(x), \end{cases} \tag{3.2}$$

where

$$\mathscr{L}_\sigma^\alpha u(t,x) = \int_{\mathbb{R}^d} \Big(u(t, x + \sigma(x,z)) - u(t,x) - \mathbb{1}_{\alpha \geqslant 1} \sigma(x,z) \cdot \nabla u(t,x) \Big) \nu_\alpha(\mathrm{d}z).$$

- By Itô formula, $s \to u(t-s, X_s^x)$ is a martingale for $s \in [0,t]$. Then

$$P_t^{\sigma,b}\phi(x) = \mathbb{E}(\phi(X_t^x)) = \mathbb{E}(u(t-s, X_s^x)) = \mathbb{E}(u(t,x)) = u(t,x).$$

Characteristic line

- Let θ_t^y be a solution of following ODE

$$\begin{cases} d\theta_t^y = -b(\theta_t^y), \\ \theta_0^y = y, \end{cases}$$

for $t \in [0, T]$ and $y \in \mathbb{R}^d$.

Remark 11

Under the condition $\boldsymbol{H}_\beta^b$, there is a constant C such that for any $|x - y| \geqslant 1$,

$$|b(x) - b(y)| \leqslant C|x - y|,$$

which implies that θ_t^y would never blow up. See Wang-Zhang[1].

[1] Degenerate SDE with Hölder-Dini drift and non-Lipschitz coefficient. SIAM J. Math. Anal. 48 (2016), 2189–2226.

Perturbation

- Define $\Theta_t^y g(x) := g(x + \theta_t^y)$. Then $\Theta_t^y u$ satisfies a new PDE

$$\begin{cases} \partial_t \Theta_t^y u(t,x) = \mathscr{L}_0^\alpha \Theta_t^y u(t,x) + \tilde{\mathscr{L}}^\alpha \Theta_t^y u(t,x) + \tilde{b}(x) \cdot \nabla \Theta_t^y u(t,x), \\ \Theta_t^y u(0,x) = \phi(x+y), \end{cases} \tag{3.3}$$

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- where $\tilde{b}(x) = \Theta_t^y b(x) - \Theta_t^y b(0)$,

$$\mathscr{L}_0^\alpha g(x) = \int_{\mathbb{R}^d} \Big(g(x+\sigma(\theta_t^y, z)) - g(x) - \mathbb{1}_{\alpha \geqslant 1} \sigma(\theta_t^y, z) \cdot \nabla g(x) \Big) \nu(\mathrm{d}z),$$

$$\tilde{\mathscr{L}}^\alpha g(x) = \int_{\mathbb{R}^d} \mathscr{D}_z^y g(x) \nu(\mathrm{d}z)$$

$$:= \int_{\mathbb{R}^d} \Big(g(x+\sigma(x+\theta_t^y, z)) - g(x+\sigma(\theta_t^y, z)) - \mathbb{1}_{\alpha \geqslant 1} \tilde{\sigma}(x,z) \cdot \nabla g(x) \Big) \nu(\mathrm{d}z),$$

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Perturbation

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with $\tilde{\sigma}(x,z) = \sigma(x+\theta_t^y, z) - \sigma(\theta_t^y, z)$.

- Notice that there is a constant C such that $|\tilde{b}(x)| \leqslant C|x|^\beta \wedge |x|$ and

$$\tilde{\sigma}(0,z) = 0, \quad |\tilde{\sigma}(x,z)| \leqslant c_0 |x||z|, \quad |\nabla_x \tilde{\sigma}(x,z)| \leqslant c_0 |z|.$$

- Notice that $\mathscr{L}_0^\alpha$ is the infinitesimal generation of the process
$$L_{s,t}^0 = \int_s^t \int_{\mathbb{R}^d} \sigma(\theta_r^y, z)\tilde{N}(\mathrm{d}r, \mathrm{d}z).$$

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- We denote by $p_{s,t}(x)$ the transition probability of $L_{s,t}^0$, then crucial lemma is available for $p_{s,t}$. By the Duhamel's formula,

$$\begin{aligned}\Theta_t^y u(t, w) = & \int_0^t \int_{\mathbb{R}^d} p_{s,t}(w - x)\tilde{\mathscr{L}}^\alpha \Theta_t^y u(s, x)\mathrm{d}x\mathrm{d}s \\ & + \int_0^t \int_{\mathbb{R}^d} p_{s,t}(w - x)\tilde{b}(x) \cdot \nabla\Theta_t^y u(s, x)\mathrm{d}x\mathrm{d}s \\ & + \int_{\mathbb{R}^d} p_{0,t}(w - x)\phi(x + y)\mathrm{d}x.\end{aligned}$$

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$$\begin{aligned}\Theta_t^y u(t,w) = &\int_0^t \int_{\mathbb{R}^d} p_{s,t}(w-x)\tilde{\mathscr{L}}^\alpha \Theta_t^y u(s,x)\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}^d} p_{s,t}(w-x)\tilde{b}(x)\cdot\nabla\Theta_t^y u(s,x)\mathrm{d}x\mathrm{d}s \\ &+ \int_{\mathbb{R}^d} p_{0,t}(w-x)\phi(x+y)\mathrm{d}x.\end{aligned}$$
- We operate the block operator Δ_j on both sides and let $w=0$,
$$\begin{aligned}&\Delta_j u(t,\theta_t^y) = \Delta_j\Theta_t^y u(t,0) = \int_0^t \int_{\mathbb{R}^d} \Delta_j p_{s,t}(-x)\tilde{\mathscr{L}}^\alpha \Theta_t^y u(s,x)\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}^d} \Delta_j p_{s,t}(-x)\tilde{b}(x)\cdot\nabla\Theta_t^y u(s,x)\mathrm{d}x\mathrm{d}s + \int_{\mathbb{R}^d} \Delta_j p_{0,t}(-x)\phi(x+y)\mathrm{d}x, \\ &:= \mathscr{I}_1^j + \mathscr{I}_2^j + \mathscr{I}_3^j.\end{aligned} \tag{3.4}$$

- If there is a constant C such that all j and $t \in (0, T]$,

$$|\mathscr{I}_1^j| \vee |\mathscr{I}_2^j| \leqslant C 2^{-\gamma_1 j} \int_0^t (t-s)^{\frac{\gamma_1}{\alpha}} \|u(s)\|_{B^{\gamma_1}_{\infty,\infty}} ds, \tag{3.5}$$

and

$$|\mathscr{I}_3^j| \leqslant C 2^{-\gamma_1 j} t^{-\frac{1}{\alpha}\left(\frac{d}{p} - \gamma_2 + \gamma_1\right)} \|\phi\|_{B^{\gamma_2}_{p,\infty}}. \tag{3.6}$$

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- Then we have

$$2^{\gamma_1 j} |\Delta_j u(t, \theta_t^y)| \leqslant C \int_0^t (t-s)^{\frac{\gamma_1}{\alpha}} \|u(s)\|_{B^{\gamma_1}_{\infty,\infty}} ds + Ct^{-\frac{1}{\alpha}\left(\frac{d}{p} - \gamma_2 + \gamma_1\right)} \|\phi\|_{B^{\gamma_2}_{p,\infty}}.$$

- If there is a constant C such that all j and $t \in (0, T]$,

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- In fact, for any $t \in [0, T]$ and $x \in \mathbb{R}^d$ there is a characteristic line $\theta^y_\cdot$ such that $\theta_t^y = x$. See H.-Wu-Zhang[2]. Therefore we take supremum of x and j,

$$\|u(t)\|_{B^{\gamma_1}_{\infty,\infty}} \leqslant C \int_0^t (t-s)^{\frac{\gamma_1}{\alpha}} \|u(s)\|_{B^{\gamma_1}_{\infty,\infty}} ds + C t^{-\frac{1}{\alpha}\left(\frac{d}{p} - \gamma_2 + \gamma_1\right)} \|\phi\|_{B^{\gamma_2}_{p,\infty}}. \tag{3.7}$$

[2] Schauder's estimate for nonlocal kinetic equations and its applications. Available at arXiv:1903.09967..

Volterra-type Gronwall inequality

Lemma 12 (Volterra-type Gronwall inequality)

Assume $A > 0$. *For any* $\theta, \vartheta > -1$ *and* $T > 0$, *there exists a constant* $C = C(A, \theta, \vartheta, T) \geqslant 0$ *such that if locally integrable functions* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfy*

$$f(t) \leqslant A \int_0^t (t-s)^\theta f(s)\mathrm{d}s + At^\vartheta, \quad t \in (0, T],$$

then

$$f(t) \leqslant Ct^\vartheta, \quad t \in (0, T].$$

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$$f(t) \leqslant A \int_0^t (t-s)^\theta f(s) \mathrm{d}s + At^\vartheta, \quad t \in (0, T],$$

then

$$f(t) \leqslant Ct^\vartheta, \quad t \in (0, T].$$

- When $\frac{d}{p} - \gamma_2 + \gamma_1 < \alpha$, $t^{-\frac{1}{\alpha}\left(\frac{d}{p} - \gamma_2 + \gamma_1\right)}$ is a local integral function on $[0, T]$. Combining Volterra-type Gronwall inequality with (3.7), we obtain main result for $\gamma_1 \in [0, \alpha)$ and $\frac{d}{p} - \gamma_2 < \alpha - \gamma_1$.
- To prove (3.5) and (3.6), we need the crucial lemma.
- To lift the limitation of γ_1 from $[0, \alpha)$ to $[0, \alpha + \alpha \wedge 1)$, we need a lift theorem by the semigroup property of Feller process.

Estimate for $\mathscr{I}_3^j$

- ($\tilde{\Delta}_j \Delta_j = \Delta_j$ and Δ_j is symmetric)$\Rightarrow$

$$\mathscr{I}_3^j = \int_{\mathbb{R}^d} \Delta_j p_{0,t}(-x)\phi(x+y)dx = \int_{\mathbb{R}^d} \Delta_j p_{0,t}(-x)\tilde{\Delta}_j \phi(x+y)dx.$$

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- (Hölder inequlity)$\Rightarrow$

$$|\mathscr{I}_3^j| \leqslant \int_{\mathbb{R}^d} |\Delta_j p_{0,t}(-x)||\tilde{\Delta}_j \phi(x+y)|dx \leqslant \|\Delta_j p_{0,t}\|_{L^q} \|\tilde{\Delta}_j \phi\|_{L^p},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

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  where $\frac{1}{p}+\frac{1}{q}=1$.
- (Definition of Besov space and crucial lemma 10)$\Rightarrow$

$$|\mathscr{I}_3^j| \leqslant 2^{-\gamma_2 j}\|\Delta_j p_{0,t}\|_{L^q}\|\phi\|_{B^{\gamma_2}_{p,\infty}} \lesssim 2^{-\gamma_1 j}t^{-\frac{1}{\alpha}(\frac{d}{p}-\gamma_2+\gamma_1)}\|\phi\|_{B^{\gamma_2}_{p,\infty}}.$$

- Notice that $\frac{d}{p}-\gamma_2+\gamma_1 \geqslant 0$, which is $\gamma_2 \leqslant \frac{d}{p}+\gamma_1$.

Estimate for $\mathscr{I}_2^j$

- Define function $\chi \in C_0^\infty$ with

$$\chi(x) = \begin{cases} 1 & \text{when } |x| < \frac{1}{2} \\ 0 & \text{when } |x| > 1. \end{cases}$$

Lemma 13

Under condition $\boldsymbol{H}_\beta^b$, function $b_z(x) := \chi(x)\Big(b(x+z) - b(z)\Big) \in C^\beta(\mathbb{R}^d)$. There is a constant C such that all $z \in \mathbb{R}^d$

$$\|b_z\|_{C^\beta} \leqslant C.$$

- By Lemma13 and the fact that

$$\|f\|_{C^\beta(\mathbb{R}^d)} \lesssim \sup_{z\in\mathbb{R}^d} \|f\|_{C^\beta(B(z,1))},$$

we assume $b \in C^\beta$ and have a commutator estimate:

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we assume $b \in C^\beta$ and have a commutator estimate:

Lemma 14 (Chen-Zhang-Zhao 2017)

For $\beta \in (0,1)$ and $\theta \in (-\beta, 0]$, there is a constant C such that

$$\|[\Delta_j, f]g\|_\infty \leqslant C2^{-j(\beta+\theta)}\|f\|_{C^\beta}\|g\|_{B^\theta_\infty},$$

where $[\Delta_j, f]g := \Delta_j fg - f\Delta_j g$.

Estimate for $\mathscr{I}_2^j$

Lemma 15

Assume $\alpha \in (\frac{1}{2}, 2)$. *Under condition* $\boldsymbol{H}_\beta^b$ *with* $\beta \in ((1-\alpha) \vee 0, \alpha \wedge 1)$. *For any* $\gamma_1 \in (0, \alpha)$ *and* $T > 0$, *there is a constant* C *such that for all* $t \in (0, T]$, $j \in \mathbb{N}_0$ *and all classical solution* u,

$$|\mathscr{I}_2^j| \leqslant C 2^{-\gamma_1} \int_0^t (t-s)^{-\frac{2\gamma_1+\beta-1}{\alpha}} \|u(s)\|_{C^{\gamma_1}} \, \mathrm{d}s.$$

Estimate for $\mathscr{I}_2^j$

Lemma 15

Assume $\alpha \in (\frac{1}{2}, 2)$. *Under condition* $\boldsymbol{H}_\beta^b$ *with* $\beta \in ((1-\alpha)\vee 0, \alpha \wedge 1)$. *For any* $\gamma_1 \in (0,\alpha)$ *and* $T > 0$, *there is a constant* C *such that for all* $t \in (0,T]$, $j \in \mathbb{N}_0$ *and all classical solution* u,

$$|\mathscr{I}_2^j| \leqslant C2^{-\gamma_1} \int_0^t (t-s)^{-\frac{2\gamma_1+\beta-1}{\alpha}} \|u(s)\|_{C^{\gamma_1}} \mathrm{d}s.$$

Proof.

Notice that

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^d} \Delta_j p_{s,t}(-x)\tilde{b}(x) \cdot \nabla \Theta_t^y u(s,x) \mathrm{d}x\mathrm{d}s \\ =& \int_0^t \int_{\mathbb{R}^d} \Delta_j p_{s,t}(-x)[\tilde{\Delta}_j, \tilde{b}(x)] \nabla \Theta_t^y u(s,x) \mathrm{d}x\mathrm{d}s \\ +& \int_0^t \int_{\mathbb{R}^d} \Delta_j p_{s,t}(-x)\tilde{b}(x) \cdot \tilde{\Delta}_j \nabla \Theta_t^y u(s,x) \mathrm{d}x\mathrm{d}s. \end{aligned}$$

By crucial lemma and commutator estimate, we complete the proof. □

Estimate for $\mathscr{I}_1^j$

- Recall that

$$\mathscr{D}_z^y f(x) = f(x+\sigma(x+\theta_t^y, z)) - f(x+\sigma(\theta_t^y, z)) - \mathbb{1}_{\alpha\geqslant 1}\tilde{\sigma}(x,z)\cdot\nabla f(x).$$

- Define

$$\mu_\theta(h) := \int_{\mathbb{R}^d} (1\wedge|x|)^\theta |h(x)| dx \quad \text{and} \quad \langle f, g\rangle := \int_{\mathbb{R}^d} f(x)g(x)dx.$$

Estimate for $\mathscr{I}_1^j$

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Lemma 16

For any $\theta \in [0,1]$, there exists a constant $C = C(d,\theta) > 0$ such that for all $|z| \leqslant \frac{1}{2c_0}$, $f \in C^\theta$ and $g \in C^2$

$$|\langle \mathscr{D}_z^y f, g\rangle| \leqslant C|z|^\theta \|f\|_\infty \left[\mu_0(|g|) + \mu_\theta(|\nabla g|)^\theta \mu_\theta(|g|)^{1-\theta}\right]$$

when $\alpha < 1$ and

$$|\langle \mathscr{D}_z^y f, g\rangle| \leqslant C|z|^{1+\theta} \|f\|_{\mathbf{C}^\theta} \left[\mu_0(|g|) + \mu_1(|\nabla g|) + \mu_{1+\theta}(|\nabla^2 g|)^\theta \mu_{1+\theta}(|\nabla g|)^{1-\theta}\right]$$

when $\alpha \geqslant 1$.

The key point of the proof

- For simplicity, we assume $\alpha < 1$ and $\phi_z(x) = \sigma(x + \theta_t^y, z)$. Rewrite

$$\mathscr{D}_z f(x) := \mathscr{D}_z^y f(x) = f(x + \phi_z(x)) - f(x + \phi_z(0)).$$

- We can let $\bar{f}(x) = f(x + \phi_z(0))$. Their C^θ norms are the same. Therefore we assume that $\phi_z(0) = 0$ and there is a constant such that $|\phi_z(x)| \leqslant C(|x| \wedge 1)|z|$.

The key point of the proof

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- We can let $\bar{f}(x) = f(x + \phi_z(0))$. Their C^θ norms are the same. Therefore we assume that $\phi_z(0) = 0$ and there is a constant such that $|\phi_z(x)| \leqslant C(|x| \wedge 1)|z|$.
- Let $\Gamma_z(x) = x + \phi_z(x)$. By change of variable, we have

$$\langle \mathscr{D}_z f, g \rangle = \langle f, \mathscr{D}_z^* g \rangle,$$

where

$$\mathscr{D}_z^* g(x) = \det(\nabla_x \Gamma_z^{-1}(x)) g(\Gamma_z^{-1}(x)) - g(x).$$

- Noticing that

$$|\det(\nabla_x \Gamma_z^{-1}(x)) - 1| \leqslant |z|, \quad \text{and} \quad |\Gamma_z^{-1}(x) - x| \leqslant CC(|x| \wedge 1)|z|,$$

we complete the proof.

Lemma 17

Let $\varepsilon \in (0, \alpha \wedge 1)$ *and* $\theta \in ((\alpha-1) \vee 0, \alpha \wedge 1)$*.For any* $\gamma \in (0, \alpha - \varepsilon)$*, there is a constant* $C > 0$ *such that for all* $j \in \mathbb{N}_0$ *and* $t \in (0, T]$*,*

$$|\mathscr{I}_1^j| \leqslant C 2^{-\gamma j} \int_0^t (t-s)^{-(\gamma+\varepsilon)/\alpha} \|u(s)\|_{C^\theta} \mathrm{d}s.$$

- Recall

$$\mathscr{I}_1^j = \int_0^t \int_{\mathbb{R}^d} \Delta_j p_{s,t}(-x) \tilde{\mathscr{L}}^\alpha \Theta_t^y u(s,x) \mathrm{d}x \mathrm{d}s.$$

Proof.

Let $\delta = \frac{\kappa}{c_0}$.We only prove the estimate for $\alpha \in (1,2)$. The case $\alpha \in (0,1]$ is similar and easier.Since the time variable and y does not play any essential role, below we drop the time variable and Θ_t^y for simplicity of notations. By definition we can make the following decomposition:

$$\tilde{\mathscr{L}}^\alpha u = \mathscr{A}_\delta u + \bar{\mathscr{A}_\delta} u,$$

where

$$\mathscr{A}_\delta u(x) = \int_{|z| \leqslant \delta} \mathscr{D}_z u(x) \nu(\mathrm{d}z) \quad \text{and} \quad \bar{\mathscr{A}_\delta} u(x) = \int_{|z| > \delta} \mathscr{D}_z u(x) \nu(\mathrm{d}z).$$

□

Proof.

$$\mathscr{I}_1^j = \int_0^t \langle \Delta_j p_{s,t}, \mathscr{A}_\delta u\rangle ds + \int_0^t \langle \Delta_j p_{s,t}, \bar{\mathscr{A}_\delta} u\rangle ds.$$

By Lemma16, we have

$$|\langle \Delta_j p_{s,t}, \mathscr{A}_\delta u\rangle| \leqslant C \int_{|z|\leqslant\delta} |z|^{1+\theta} \nu(\mathrm{d}z) \|u(s)\|_{\mathbf{C}^\theta} \mathscr{B}(s,t),$$

where

$$\mathscr{B}(s,t) = \left| \sum_{i=0}^{1} \mu_i(|\nabla^i \Delta_j p_{s,t}|) + \mu_{1+\theta}(|\nabla^2 \Delta_j p_{s,t}|)^\theta \mu_{1+\theta}(|\nabla \Delta_j p_{s,t}|)^{1-\theta} \right|.$$

Let $\alpha < 1+\theta < \alpha + \frac{\varepsilon}{2}$. By crucial lemma, we obtain that

$$| \int_0^t \langle \Delta_j p_{s,t}, \mathscr{A}_\delta u\rangle \mathrm{d}s| \leqslant C \int_0^t \|u(s)\|_{\mathbf{C}^\theta} \mathscr{B}(s,t) ds$$

$$\lesssim 2^{-\gamma j} \int_0^t (t-s)^{-\frac{\gamma}{\alpha}} \|u(s)\|_{C^\theta} \mathrm{d}s + 2^{-(\gamma-\varepsilon)j} \int_0^t (t-s)^{-\frac{\gamma}{\alpha}} \|u(s)\|_{C^\theta} \mathrm{d}s,$$

where

$$\int_0^t \mu_{1+\theta}(|\nabla^j \Delta_j p_{s,t}|) \|u(s)\|_{C^\theta} \mathrm{d}s \lesssim \int_0^t \int_{\mathbb{R}^d} |x|^{\alpha-\frac{\varepsilon}{2}} |\nabla^j \Delta_j p_{s,t}(x)| \|u(s)\|_{C^\theta} \mathrm{d}s,$$

□

Proof.

so

$$\int_0^t \mu_{1+\theta}(|\nabla^2\Delta_j p_{s,t}|)^\theta \mu_{1+\theta}(|\nabla\Delta_j p_{s,t}|)^{1-\theta}\|u(s)\|_{C^\theta}\mathrm{d}s$$

$$\lesssim 2^{-(\alpha-\frac{\varepsilon}{2}-1-\theta+\gamma)j}\int_0^t (t-s)^{-\frac{\gamma}{\alpha}}\|u(s)\|_{C^\theta}\mathrm{d}s \leqslant 2^{-\gamma j}\int_0^t (t-s)^{-\frac{\gamma}{\alpha}}\|u(s)\|_{C^\theta}\mathrm{d}s.$$

For $\bar{\mathscr{A}}_\delta u$, by Fubini's theorem and the integration by parts, we have

$$|\langle\Delta_j p_{s,t}, \bar{\mathscr{A}}_\delta u\rangle| \leqslant \int_{|z|>\delta}\int_{\mathbb{R}^d}|\Delta_j p_{s,t}(x)||u(x+\sigma(x,z)) - u(x+\sigma(0,z))|\mathrm{d}x\mathrm{d}z$$

$$+\int_{|z|>\delta}\int_{\mathbb{R}^d}\left|\Big(\Delta_j p_{s,t}(x)\mathrm{div}_x\sigma(x,z) + (\sigma(x,z)-\sigma(0,z))\cdot\nabla\Delta_j p_{s,t}(x)\Big)u(x)\right|\mathrm{d}x\mathrm{d}z$$

$$\leqslant \|u(s)\|_\infty\Big(\mu_0(|\Delta_j p_{s,t}|) + \mu_1(|\nabla\Delta_j p_{s,t}|)\int_{|z|>\delta}|z|\mathrm{d}z\Big).$$

By crucial lemma again, we obtain that

$$\int_0^t |\langle\Delta_j p_{s,t}, \bar{\mathscr{A}}_\delta u\rangle|\mathrm{d}s \lesssim 2^{-\gamma j}\int_0^t (t-s)^{\frac{\gamma}{\alpha}}\|u(s)\|_\infty\mathrm{d}s.$$

□

Lift lemma

Lemma 18

Assume one of the following conditions holds,

- $\alpha \in (0,2)$, $b \equiv 0$ *and let* $\beta = 1$.
- $\alpha \in (\frac{1}{2}, 2)$ *and condition* $(\mathbf{H}^b_\beta)$ *holds with* $\beta \in ((1-\alpha) \vee 0, \alpha \wedge 1)$.

Under condition $(\mathbf{H}^\sigma_\mu)$*, for any*

$$\gamma \in (\alpha, \alpha + \alpha \wedge \beta), \quad \delta \in [0, \alpha),$$

there is a constant C_T *such that for all* $\phi \in C_0^\infty(\mathbb{R}^d)$ *and all* $t \in (0, T]$,

$$\|P_t^{\sigma,b}\phi\|_{B^\gamma_{\infty,\infty}} \leqslant C_T t^{-\frac{\delta}{\alpha}} \|\phi\|_{B^{\gamma-\delta}_{p,\infty}}. \tag{3.8}$$

- Notice that $P_t^{\sigma,b}\phi = P_{\frac{t}{2}}^{\sigma,b} P_{\frac{t}{2}}^{\sigma,b}\phi$ and $(\alpha, \alpha + \alpha \wedge \beta) - \alpha \subset (0, \alpha)$, by this C-K property, we obtain the main result.

Future works

- We prove that the solution of SDE driven by cylindrical Lévy process has a density in Sobolev space $H^{s,r}$ with

$$s < \alpha - (\alpha - 1) \vee (1 - \beta) \quad \text{and} \quad r < \frac{d}{d - \alpha + s + \beta - 1},$$

but this result does not imply that this density is continuous. So how to improve the index s and how to make r greater are interesting.

- In our work, we only consider the strong Feller property, which only depend on the distribution of X_t^x. Moreover, the continuous property of σ is enough to guarantee the existence of weak solution. So how to drop the assumption that σ is Lipschitz.

Thanks for your attention!