Heat kernel of nonlocal kinetic operators

Zimo Hao[1]
郝子墨

Based on a joint work with Xicheng Zhang[1]

[1] Wuhan University

Peking University-2019

School of Mathematical Sciences, Peking University

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Outline

- Introduction
- Main results
- From non-local operator to process
- Our approach
- Some techniques

Part 1 : Introduction

Motivation

- Suppose a space $X = \mathbb{R}^d$ is full of gas. The gas is observed on a time interval $[0, T]$, and $V = \mathbb{R}^d$ is the tangent space of X standing for the velocity of the gas.
- For any fixed time t, the quantity $f(t, x, \mathrm{v})\mathrm{d}x\mathrm{d}\mathrm{v}$ stands for the quantity of particles in the volume element $\mathrm{d}x\mathrm{d}\mathrm{v}$ centered at (x, v). Obviously, f is a non-negative function and we assume that it is very nice.
- Firstly, we assume that there is no collision among the gas and each particle travels at constant velocity, along a straight line. Then we have the following invariance alone the characteristic line:

$$f(t, x, \mathrm{v}) = f(0, x - t\mathrm{v}, \mathrm{v}).$$

  In other words, f is a solution to the following transport equation

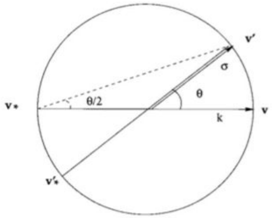
$$\partial_t f + \mathrm{v} \cdot \nabla_x f = 0.$$

- However, particle is not ghost, there are many collisions among the gas.

- We assume that the gas is dilute enough that the effect of interactions involving more than two particles can be neglected, which means that we only consider binary collisions. Furthermore, we assume that collision is elastic, which is the following meaning

$$\begin{cases} \mathrm{v}' + \mathrm{v}'_* = \mathrm{v} + \mathrm{v}_*, \\ |\mathrm{v}'|^2 + |\mathrm{v}'_*|^2 = |\mathrm{v}|^2 + |\mathrm{v}_*|^2, \end{cases} \tag{1.1}$$

where v, v_* stand for the velocities before collision, and $\mathrm{v}', \mathrm{v}'_*$ stand for the velocities after collision.



- Notice that there are $d+1$ equations but $2d$ unknowns. Therefore, the solution of the velocities after collision has $d-1$ degrees of freedom. Actually, there is a $\sigma \in \mathbb{S}^{d-1}$ such that the solution of (1.1) have the following σ-representation:

$$\begin{pmatrix} \mathrm{v}' \\ \mathrm{v}'_* \end{pmatrix} = \begin{pmatrix} \mathbb{I} - \sigma \otimes \sigma & \sigma \otimes \sigma \\ \sigma \otimes \sigma & \mathbb{I} - \sigma \otimes \sigma \end{pmatrix} \begin{pmatrix} \mathrm{v} \\ \mathrm{v}_* \end{pmatrix},$$

where $\sigma \otimes \sigma \mathrm{v} := \langle \mathrm{v}, \sigma \rangle \sigma$.

- Under the above assumption and other assumptions(see [1]), in 1872 Boltzmann was able to derive a quadratic collision operator Q which accurately models the effect of interactions on the f:

$$\partial_t f(t,x,\mathrm{v}) + \mathrm{v}\cdot\nabla_x f(t,x,\mathrm{v}) = Q(f,f)(t,x,\mathrm{v}), \qquad (1.2)$$

  where

$$Q(f,g)(\mathrm{v}) := \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \Big(f(\mathrm{v}'_*)g(\mathrm{v}') - f(\mathrm{v}_*)g(\mathrm{v})\Big) B(\mathrm{v}-\mathrm{v}_*,\sigma)\mathrm{d}\sigma\mathrm{d}\mathrm{v},$$

  where $B:\mathbb{R}^d\times\mathbb{S}^{d-1}\to\mathbb{R}$ is a non-negative function called collision kernel and defined as

$$B(\mathrm{v},\sigma) := |\mathrm{v}|^\gamma b(cos\theta)$$

  where $cos\theta := |\langle \mathrm{v},\sigma\rangle|/|\mathrm{v}|$ and b,γ is related to the property of the gas with

$$b(s)\asymp s^{-1-\alpha}, \alpha\in(0,2), \gamma\in\begin{cases}[-\alpha,+\infty) & \text{when the gas has hard potentials,}\\ (-d,-\alpha) & \text{when the gas has soft potentials.}\end{cases}.$$

- The equation (1.2) is called Boltzmann equation.

[1] C. Villani: A review of mathematical topics in collisional kinetic theory. Handbook of Fluid Mechanics. Ed. S. Friedlander, D.Serre, 2002

- By an elementary calculation(see [1]), the collision operator has the following Carleman's representation:

$$Q(f,g)(\mathrm{v}) := 2\int_{\mathbb{R}^d}\int_{\{h\cdot w=0\}}\Big[f(\mathrm{v}-h)g(\mathrm{v}+w) - f(\mathrm{v}-h+w)g(\mathrm{v})\Big] \times B(h-w, w/|w|)|w|^{1-d}\mathrm{d}h\mathrm{d}w.$$

- When $b(s) = s^{-1-\alpha}$, we have the following decomposition:

$$Q(f,g) = g(\mathrm{v})H_f(\mathrm{v}) + L(f,g),$$

where

$$H_f(\mathrm{v}) = 2\int_{\mathbb{R}^d}\int_{\{h\cdot w=0\}}\Big(f(\mathrm{v}-h) - f(\mathrm{v}-h+w)\Big)\frac{|h-w|^{\gamma+1+\alpha}}{|w|^{\alpha+d}}\mathrm{d}h\mathrm{d}w,$$

and

$$L(f,g) = \int_{\mathbb{R}^d}\Big(g(\mathrm{v}+w) - g(\mathrm{v})\Big)\frac{K_f(\mathrm{v},w)}{|w|^{\alpha+d}}\mathrm{d}w$$

with

$$K_f(\mathrm{v},w) := 2\int_{\{h\cdot w=0\}} f(\mathrm{v}-h)|h-w|^{\gamma+1+\alpha}\mathrm{d}h.$$

[1] Z.-Q. Chen and X. Zhang, L^p-maximal hypoelliptic regularity of nonlocal kinetic Fokker-Planck operators. J. Math. Pures Appl. (9), 116 (2018), 52-87

- We linearize the Boltzmann equation and get a equation involves non-local operator of fractional Laplacian type:

$$\partial_t g + \mathrm{v} \cdot \nabla_x g = \mathrm{p.v.} \int_{\mathbb{R}^d} \Big(g(\mathrm{v}+w) - g(\mathrm{v}) \Big) \frac{K_f(\mathrm{v}, w)}{|w|^{d+\alpha}} \mathrm{d}w + gH_f .$$

- Notice that if $K_f \equiv C$,

$$\mathrm{p.v.} \int_{\mathbb{R}^d} \Big(g(\mathrm{v}+w) - g(\mathrm{v}) \Big) \frac{K_f(\mathrm{v}, w)}{|w|^{d+\alpha}} \mathrm{d}w = -(-\Delta)^{\frac{\alpha}{2}} g(\mathrm{v}).$$

- Therefore, we regard $\mathrm{p.v.} \int_{\mathbb{R}^d} \Big(g(\mathrm{v}+w) - g(\mathrm{v}) \Big) \frac{K_f(\mathrm{v},w)}{|w|^{d+\alpha}} \mathrm{d}w$ as a α order term and gH_f as a zero order term in g. Hence, we neglect the term gH_f and consider a generality PDE.

Our model

- In our work, we assume $\alpha \in (0,2)$ and consider the following non-local PDE:

$$\begin{cases} \partial_t u(t,x,\mathrm{v}) + \mathscr{L}^{(\alpha)}_{\kappa,\mathrm{v}} u(t,x,\mathrm{v}) + \mathrm{v}\cdot\nabla_x u(t,x,\mathrm{v}) = f(t,x,\mathrm{v}),\ t \geqslant 0, \\ u(0,x,\mathrm{v}) = \varphi(x,\mathrm{v}), \end{cases} \tag{1.3}$$

  where

$$\mathscr{L}^{(\alpha)}_{\kappa,\mathrm{v}} u(x,\mathrm{v}) := \text{p.v.} \int_{\mathbb{R}^d} \Big(u(x,\mathrm{v}+z) - u(x,\mathrm{v}) \Big) \kappa(t,x,\mathrm{v},z)\nu(\mathrm{d}z),$$

  with some measurable coefficient $\kappa : \mathbb{R}_+ \times \mathbb{R}^{3d} \to \mathbb{R}_+$ and measure ν in the form of

$$\nu(A) := \int_0^\infty \frac{\mathrm{d}r}{r^{1+\alpha}} \int_{\mathbb{S}^{d-1}} \mathbf{1}_A(r\omega)\mu(\mathrm{d}\omega),$$

  where μ is a symmetric finite measure in $\mathbb{S}^{d-1}$ with the following non-degenerate property

$$\inf_{e\in\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |\langle e,\omega\rangle| \mu(\mathrm{d}\omega) > 0.$$

- Question: Existence and Uniqueness.

Scaling

- Let us consider the elliptic equation:

$$(-\Delta_{\mathrm{v}})^{\frac{\alpha}{2}} u + \mathrm{v} \cdot \nabla_x u = 0.$$

- We suppose $u_\varepsilon(x, \mathrm{v}) = u(\varepsilon^\beta x, \varepsilon \mathrm{v})$ still satisfy the equation, then we have

$$\left.\begin{array}{l} (-\Delta_{\mathrm{v}})^{\frac{\alpha}{2}} u_\varepsilon(x, \mathrm{v}) = \varepsilon^\alpha (-\Delta_{\mathrm{v}})^{\frac{\alpha}{2}} u(\varepsilon^\beta x, \varepsilon \mathrm{v}) \\ \mathrm{v} \cdot \nabla_x u_\varepsilon(x, \mathrm{v}) = \varepsilon^\beta \mathrm{v} \cdot \nabla_x u(\varepsilon^\beta x, \varepsilon \mathrm{v}) \\ \varepsilon^\alpha \Big((-\Delta_{\mathrm{v}})^{\frac{\alpha}{2}} u(\varepsilon^\beta x, \varepsilon \mathrm{v}) + \varepsilon \mathrm{v} \cdot \nabla_x u(\varepsilon^\beta x, \varepsilon \mathrm{v}) \Big) = 0 \end{array} \right\} \Rightarrow \beta = \alpha + 1.$$

- Notice that $\nabla_x \approx \nabla_{\mathrm{v}}^{\alpha+1}$ is very singular.
- Naturally, we introduce the following distance in $\mathbb{R}^{2d}$ by

$$|z_1 - z_2|_a := |x_1 - x_2|^{\frac{1}{1+\alpha}} + |\mathrm{v}_1 - \mathrm{v}_2| \quad \text{where } z_i = (x_i, \mathrm{v}_i) \in \mathbb{R}^d \times \mathbb{R}^d.$$

- Moreover, we introduce the anisotropic Hölder-Zygmund space

$$\mathbf{C}^s_a := \Big\{ f \in \mathbb{R}^{2d} \to \mathbb{R} : \|f\|_{\mathbf{C}^s_a} := \|f\|_\infty + [f]_{\mathbf{C}^s_a} < \infty \Big\},$$

where

$$[f]_{\mathbf{C}^s_a} := \sup_h \|\delta_h^{[s]+1} f\|_\infty / |h|_a^s, \quad \delta_h f(z) := f(z+h) - f(z).$$

Assumption

- We introduce the Hölder-Zygmund space in $\mathbb{R}^d_x$ and $\mathbb{R}^d_{\rm v}$ first,

$$\mathbf{C}^s_x := \Big\{ f \in \mathbb{R}^{2d} \to \mathbb{R} : \|f\|_{\mathbf{C}^s_a} := \|f\|_\infty + [f]_{\mathbf{C}^s_a} < \infty \Big\},$$

where

$$[f]_{\mathbf{C}^s_x} := \sup_h \|\delta^{[s]+1}_{h,x} f\|_\infty / |h|^s, \quad \delta_{h,x} f(x, {\rm v}) := f(x+h, {\rm v}) - f(x, {\rm v}).$$

The space $\mathbf{C}^s_{\rm v}$ is the same.

$(\mathbf{H}^{\kappa,bdd}_{\theta,\beta})$ There are some $\beta \in (0, \alpha)$, $\theta \in [\beta, \alpha(1+\alpha))$ and $c_i > 0$ for $i = 0, 1$ such that for all $t \in \mathbb{R}_+$ and $x, {\rm v} \in \mathbb{R}^d$

$$\kappa(t, x, {\rm v}) \geqslant c_0 \quad \|\kappa(t)\|_{\mathbf{C}_x^{\frac{\theta}{1+\alpha}}} \vee \|\kappa(t)\|_{\mathbf{C}^\beta_{\rm v}} \leqslant c_1.$$

Known results

2012 (Alexandre)

When $\varphi \equiv 0$ and $\mathscr{L}^{(\alpha)}_{\kappa,\mathrm{v}} = \Delta^{\frac{\alpha}{2}}_{\mathrm{v}}$, there is a L^2-regularity:

$$\|\Delta_x^{\frac{\alpha}{2(1+\alpha)}} u\|_{L^2} + \|\Delta_{\mathrm{v}}^{\frac{\alpha}{2}} u\|_{L^2} \leqslant C\|f\|_{L^2}.$$

- This estimate comes from the Fourier's transformation.

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- This estimate comes from the Fourier's transformation.

2018 (Chen-Zhang)
Assume κ has both upper bounded and lower bounded. There is a L^p-regularity for any $p \in (1, +\infty)$:

$$\|\Delta_x^{\frac{\alpha}{2(1+\alpha)}} u\|_{L^p} + \|\Delta_{\mathrm{v}}^{\frac{\alpha}{2}} u\|_{L^p} \leqslant C_p\|f\|_{L^p}.$$

- They get this estimate by the interpolation and a crucial lemma:

$$\int_{\mathbb{R}^d} |x|^\beta |\mathrm{v}|^\gamma |\nabla_x^n \nabla_{\mathrm{v}}^m p_{s,t}(x,\mathrm{v})| \mathrm{d}x\mathrm{d}\mathrm{v} \leqslant C(t-s)^{-\frac{(1+\alpha)(m-\beta)}{\alpha} - \frac{n-\gamma}{\alpha}},$$

where $p_{s,t}(x,\mathrm{v})$ is the heat kernel of the operator $\mathscr{L}^{(\alpha)}_{\kappa,\mathrm{v}}$ when κ is independent with (x,v).

2019 (H.-Peng-Zhang)
Assume $\nu(\mathrm{d}z) = |z|^{-d-\alpha}\mathrm{d}z$, $\kappa \in C_b^\infty(\mathbb{R}^{2d})$ with lower bounded is independent with time t, and for any $i, j \in \mathbb{N}_0$ there is a $C_{i,j}$ such that for all $x, \mathrm{v} \in \mathbb{R}^d$

$$|\nabla_x^{i+1}\nabla_{\mathrm{v}}^j\kappa(x,\mathrm{v})| \leqslant \frac{C_{i,j}}{1+|\mathrm{v}|^2}.$$

Then there is a non-negative function $p_t(x, \mathrm{v}, y, w)$ such that for any $(y, w) \in \mathbb{R}^{2d}$, $p_t(\cdot, y, w) \in C_b^\infty(\mathbb{R}^{2d})$ and for any $\varphi \in C_b^1$, $\int_{\mathbb{R}^{2d}} \varphi(y, w)p_t(x, \mathrm{v}, y, w)\mathrm{d}y\mathrm{d}w$ solves the PDE (1.3) with $f \equiv 0$.

- This result comes from the Malliavin calculus.

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2019 (H.-Wu-Zhang)
Assume $\varphi \equiv 0$ and $\nu(\mathrm{d}z) = |z|^{-d-\alpha}\mathrm{d}z$. Under the condition $(\mathbf{H}_{\theta,\beta}^{\kappa,bdd})$, there is a Schauder estimate

$$\sup_{t\in[0,T]}\left(\|u(t)\|_{\mathbf{C}_x^{\frac{\alpha+\theta}{1+\alpha}}} + \|u(t)\|_{\mathbf{C}_{\mathrm{v}}^{\alpha+\beta}}\right) \leqslant C \sup_{t\in[0,T]}\left(\|f(t)\|_{\mathbf{C}_x^{\frac{\theta}{1+\alpha}}} + \|f(t)\|_{\mathbf{C}_{\mathrm{v}}^{\beta}}\right).$$

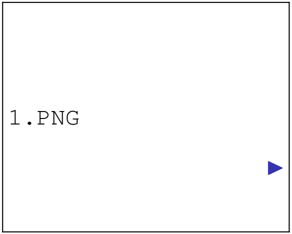
Part 2: Our main results

Anisotropic Littlewood-Paley decomposition and Besov space

- Denote by B^a_r the unit ball with radio r in metric space $(\mathbb{R}^{2d}, |\cdot|_a)$, i.e.
$$B^a_r := \{(\xi,\eta) \in \mathbb{R}^{2d}\ ;\ |(\xi,\eta)|_a < r\}.$$
Let ϕ^a_0 be a C^∞-function on $\mathbb{R}^{2d}$ with
$$\phi^a_0(\xi,\eta) = 1 \ \text{ for } \ (\xi,\eta) \in B^a_1 \ \text{ and } \ \phi^a_0(\xi,\eta) = 0 \ \text{ for } \ (\xi,\eta) \notin B^a_2.$$
- For $(\xi,\eta) \in \mathbb{R}^{2d}$ and $j \in \mathbb{N}$, define
$$\phi^a_j(\xi,\eta) := \phi^a_0(2^{-(1+\alpha)j}\xi, 2^{-j}\eta) - \phi^a_0(2^{-(1+\alpha)(j-1)}\xi, 2^{-(j-1)\eta}).$$

Anisotropic Littlewood-Paley decomposition and Besov space

- Denote by B_r^a the unit ball with radio r in metric space $(\mathbb{R}^{2d}, |\cdot|_a)$, i.e.
$$B_r^a := \{(\xi,\eta) \in \mathbb{R}^{2d}\ ;\ |(\xi,\eta)|_a < r\}.$$
Let ϕ_0^a be a C^∞-function on $\mathbb{R}^{2d}$ with
$$\phi_0^a(\xi,\eta) = 1 \ \text{ for } \ (\xi,\eta) \in B_1^a \ \text{ and } \ \phi_0^a(\xi,\eta) = 0 \ \text{ for } \ (\xi,\eta) \notin B_2^a.$$
- For $(\xi,\eta) \in \mathbb{R}^{2d}$ and $j \in \mathbb{N}$, define
$$\phi_j^a(\xi,\eta) := \phi_0^a(2^{-(1+\alpha)j}\xi, 2^{-j}\eta) - \phi_0^a(2^{-(1+\alpha)(j-1)}\xi, 2^{-(j-1)\eta}).$$



- It is easy to see that for $j \in \mathbb{N}$, $\phi_j^a(\xi,\eta) = \phi_1^a(2^{-(1+\alpha)(j-1)}\xi, 2^{-(j-1)\eta}) \geqslant 0$ and $\text{supp}\phi_j^a \subset B_{2^{j+1}}^a \backslash B_{2^{j-1}}^a$ (see the picture left),
$$\sum_{j=0}^{k} \phi_j^a(\xi,\eta) = \phi_0^a(2^{-(1+\alpha)k}\xi, 2^{-k}\eta) \to 1, \ \ k \to \infty.$$
- Notice that $\{\phi_j^a\}_{j\in\mathbb{N}_0}$ is a partition of unity of
$$\mathbb{R}^d = B_2^a \cup \left(\cup_{j\in\mathbb{N}} \left(B_{2^{j+1}}^a \setminus B_{2^{j-1}}^a\right)\right).$$

- For given $j \in \mathbb{N}_0$, the block operator Δ_j^a is defined on $\mathscr{S}'$ by

$$\Delta_j^a f(x,\mathrm{v}) := \mathscr{F}^{-1}(\phi_j^a \mathscr{F}(f))(x,\mathrm{v}) = \mathscr{F}^{-1}(\phi_j^a) * f(x,\mathrm{v})$$
$$= \int_{\mathbb{R}^{2d}} \mathscr{F}^{-1}(\phi_1^a)(y,w) f(x - 2^{-(1+\alpha)(j-1)}y, \mathrm{v} - 2^{-(j-1)}w)\mathrm{d}y\mathrm{d}w.$$

- For $j \in \mathbb{N}_0$, by definition it is easy to see that

$$\Delta_j^a = \Delta_j^a \widetilde{\Delta}_j^a, \quad \text{where } \widetilde{\Delta}_j^a := \Delta_{j-1}^a + \Delta_j^a + \Delta_{j+1}^a \text{ with } \Delta_{-1}^a \equiv 0, \tag{2.1}$$

and Δ_j^a is symmetric in the sense that

$$\langle \Delta_j^a f, g \rangle = \langle f, \Delta_j^a g \rangle.$$

- The cut-off low frequency operator S_k is defined by

$$S_k f := \sum_{j=0}^{k-1} \Delta_j^a f = \int_{\mathbb{R}^{2d}} \check{\phi}_0^a(y,w) f(x - 2^{-(1+\alpha)k}y, v - 2^{-k}w)\mathrm{d}y\mathrm{d}w \to f. \tag{2.2}$$

- We rewrite (2.2) as

$$f = \sum_{j=0}^{\infty} \Delta_j^a f,$$

which is called the Littlewood-Paley decomposition.

Definition 1 (Anistropy Besov space)

For any $s \in \mathbb{R}$ and $p \in [1, \infty]$, the anistropy Besov space $\mathbf{B}^s_{a,\infty}$ is defined by

$$\mathbf{B}^s_{a,\infty}(\mathbb{R}^d) := \Big\{ f \in \mathscr{S}'(\mathbb{R}^d) : \|f\|_{\mathbf{B}^s_{a,\infty}} := \sup_{j \geqslant 0} \big(2^{sj} \|\Delta^a_j f\|_\infty \big) < \infty \Big\}.$$

Proposition 2 (H. Triebel)

For any $s > 0$,

$$\mathbf{C}^s_a(\mathbb{R}^{2d}) = \mathbf{B}^s_{a,\infty}(\mathbb{R}^{2d}) = \mathbf{C}_x^{\frac{s}{1+\alpha}}(\mathbb{R}^{2d}) \cap \mathbf{C}^s_{\mathrm{v}}(\mathbb{R}^{2d}).$$

Moreover,

$$L^\infty(\mathbb{R}^{2d}) \subset \mathbf{B}^0_{a,\infty}(\mathbb{R}^{2d}).$$

- The proof of the proposition 2 can be found in [1].

[1] Serguei Dachkovski, Anisotropic function spaces and related semi-linear hypoelliptic equations, Math. Nachr. 248/249 (2003), 40-61.

Notation

- We introduce $L^p_x L^q_{\mathrm{v}}$ and $L^q_{\mathrm{v}} L^p_x$ norm in the form of

$$\|f\|_{L^p_x L^q_{\mathrm{v}}} := \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|f(x,\mathrm{v})|^p \mathrm{d}x\right)^{q/p}\mathrm{d}\mathrm{v}\right)^{1/q};$$

$$\|f\|_{L^q_{\mathrm{v}} L^p_x} := \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|f(x,\mathrm{v})|^q \mathrm{d}\mathrm{v}\right)^{p/q}\mathrm{d}x\right)^{1/p},$$

where

$$\int_{\mathbb{R}^d}|f(y)|^\infty \mathrm{d}y^{1/\infty} := \sup_{y\in\mathbb{R}^d}|f(y)|.$$

- Notice that $L^p_x L^q_{\mathrm{v}} \neq L^q_{\mathrm{v}} L^p_x$. In fact, when $p \geqslant q$, we have

$$\|f\|_{L^q_{\mathrm{v}} L^p_x} \leqslant \|f\|_{L^p_x L^q_{\mathrm{v}}}.$$

- Here we define a $\mathbb{L}^{p,q}$ norm:

$$\|f\|_{\mathbb{L}^{p,q}} := \|f\|_{L^q_{\mathrm{v}} L^p_x} \vee \|f\|_{L^p_x L^q_{\mathrm{v}}}.$$

Classical solution

In our work, we consider the existence and the regularity of the following classical solutions.

Definition 3

Fix $\varphi \in C_b(\mathbb{R}^{2d})$ and $f \in L^\infty(\mathbb{R}^{2d})$. We call a continuous function $u : (0,\infty) \times \mathbb{R}^{2d} \to \mathbb{R}$ with $u \in L^\infty_{loc}((0,\infty); \mathbf{C}^{1+\varepsilon}_x \cap \mathbf{C}^{\alpha\vee 1+\varepsilon}_{\mathrm{v}})$ a classical solution of PDE (1.3) if for all $t \geqslant 0$ and $x, \mathrm{v} \in \mathbb{R}^d$,

$$u(t,x,\mathrm{v}) = \varphi(x,\mathrm{v}) + \int_0^t \Big(\mathscr{L}^{(\alpha)}_{\kappa,\mathrm{v}} + \mathrm{v}\cdot\nabla_x\Big) u(r,x,\mathrm{v}) + f(r,x,\mathrm{v})\mathrm{d}r.$$

Our assumption

($\mathbf{H}^{\kappa}_{\theta,\beta}$) There are some $\beta \in (0, \alpha)$, $\theta \in [\beta, \alpha(1+\alpha))$ and $c_i > 0$ for $i = 0, 1, 2, 3$ such that for all $t \in \mathbb{R}_+$, $x, \mathrm{v}, y, w \in \mathbb{R}^d$ and $r > 0$,

$$\inf_{\omega \in \mathbb{S}^{d-1}} \int_{|z| \leqslant r} (\omega \cdot z)^2 \kappa(t, x, \mathrm{v}, z) \nu(\mathrm{d}z) \geqslant c_0 r^{2-\alpha}, \tag{2.3}$$

$$\int_{|z| \leqslant r} |z|^2 \kappa(t, x, \mathrm{v}, z) \nu(\mathrm{d}z) \leqslant c_1 r^{2-\alpha}, \tag{2.4}$$

$$\int_{|z| \leqslant r} |z|^2 |\kappa(t, x, \mathrm{v}, z) - \kappa(t, y, w, z)| \nu(\mathrm{d}z) \leqslant c_2 r^{2-\alpha} (|x-y|^{\frac{\theta}{1+\alpha}} + |\mathrm{v} - w|^{\beta}). \tag{2.5}$$

- Notice that condition ($\mathbf{H}^{\kappa,bdd}_{\theta,\beta}$) is **stronger** then condition ($\mathbf{H}^{\kappa}_{\theta,\beta}$).
- Notice that (2.3) is strictly weaker then the condition $\kappa \geqslant c_0$.
  For example, we let $\kappa(x, \mathrm{v}, z) = \mathbf{1}_{V(x,\mathrm{v})}(z)$ where

$$V(x, \mathrm{v}) = \{z \in \mathbb{R}^d \; ; \; \langle \frac{z}{|z|}, \xi(x, \mathrm{v}) \rangle < \delta\}$$

  with some measurable function $\xi : \mathbb{R}^{2d} \to \mathbb{S}^{d-1}$ and $\delta > 0$. Then this κ satisfies condition (2.3) but does **not** have lower bounded.

Main Results

Theorem 4 (Hölder estimate)

Assume $\alpha \in (0,2)$ and $f \equiv 0$. Under the condition $\mathbf{H}^{\kappa}_{\theta,\beta}$, fix any $p, q \in [1, +\infty]$ with

$$\Theta := \frac{d(\alpha+1)}{p} + \frac{d}{q} < \beta.$$

(i) *For any $\gamma \in (0, \alpha+\beta)$, there is a constant $C > 0$ such that for all classical solutions of PDE (1.3) $u(t,x,\mathrm{v})$ and all $0 < t \leqslant T$,*

$$\|u(t)\|_{\mathbf{C}^{\gamma}_{\mathrm{v}}} \leqslant C t^{-\frac{\gamma+\Theta}{\alpha}} \|\phi\|_{\mathbb{L}^{p,q}}.$$

(ii) *For any $\gamma \in (0, \alpha+\theta)$, there is a constant $C > 0$ such that for all classical solutions of PDE (1.3) $u(t,x,\mathrm{v})$ and all $0 < t \leqslant T$,*

$$\|u(t)\|_{\mathbf{C}_x^{\frac{\gamma}{1+\alpha}}} \leqslant C t^{-\frac{\gamma+\Theta}{\alpha}} \|\phi\|_{\mathbb{L}^{p,q}}.$$

- When $1 < \alpha+\beta < \alpha+\alpha$, which is that $\alpha > \frac{1}{2}$, we obtain the gradient estimate for variable v.
- When $1+\alpha < \alpha+\theta < \alpha+\alpha(1+\alpha)$, which is that $\alpha > \frac{\sqrt{5}-1}{2}$, we obtain the gradient estimate for variable x.

Main Results

Theorem 5 (Existence of the fundamental solution)

Assume $\alpha \in (\frac{\sqrt{5}-1}{2}, 2)$ *and* $\nu(\mathrm{d}z) = |z|^{-d-\alpha}\mathrm{d}z$. *Under the condition* $(\mathbf{H}^{\kappa}_{\theta,\beta})$ *with* $\theta > 1$, *for any* $0 \leqslant s < t$, *there is a non-negative function* $p^{\kappa}_{s,t}(x, \mathrm{v}, y, w)$ *in* $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ *which has the following property: for* $x, \mathrm{v} \in \mathbb{R}^d$, $p^{\kappa}_{s,t}(x, \mathrm{v}, \cdot, \cdot) \in \mathbb{L}^{p,q}$ *with*

$$p, q \in [1, +\infty], \quad \frac{1}{p'} + \frac{1}{p} = \frac{1}{q'} + \frac{1}{q} = 1 \quad \text{and} \quad \Theta' := \frac{d(\alpha+1)}{p'} + \frac{d}{q'} < \beta,$$

such that for any uniform continuous bounded function φ *in* $\mathbb{R}^{2d}$ *and continuous bounded function* f,

$$u(t, x, \mathrm{v}) = \int_{\mathbb{R}^{2d}} p^{\kappa}_{0,t}(x, \mathrm{v}, y, w)\varphi(y, w)\mathrm{d}y\mathrm{d}w + \int_0^t \int_{\mathbb{R}^{2d}} p^{\kappa}_{s,t}(x, \mathrm{v}, y, w) f(s, y, w)\mathrm{d}y\mathrm{d}w\mathrm{d}s,$$

is a classical solution of PDE (1.3) *with*

$$\|u(t) - \varphi\|_{\mathbf{B}^0_{a,\infty}} \to 0 \quad \text{as} \quad t \to 0.$$

Part 3: From operator to process

Related process

- We let $\kappa_0(t,z):\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}_+$ with $\kappa_0(t,z)=\kappa_0(t,-z)$ satisfying

$$\inf_{\omega\in\mathbb{S}^{d-1}}\int_{|z|\leqslant r}(\omega\cdot z)^2\kappa(t,x,\mathrm{v},z)\nu(\mathrm{d}z)\geqslant c_0 r^{2-\alpha},$$
$$\int_{|z|\leqslant r}|z|^2\kappa(t,x,\mathrm{v},z)\nu(\mathrm{d}z)\leqslant c_1 r^{2-\alpha}. \tag{3.1}$$

- From (3.1), we obtain that for any $\beta<\alpha<\gamma$,

$$\int_{|z|\leqslant r}|z|^\beta\kappa(t,x,\mathrm{v},z)\nu(\mathrm{d}z)=\sum_{k=0}^{\infty}\int_{2^{-k-1}r<|z|\leqslant 2^{-k}r}|z|^\beta\kappa(t,x,\mathrm{v},z)\nu(\mathrm{d}z)$$
$$\leqslant c_1 2^{2-\beta}r^{\beta-\alpha},$$
$$\int_{|z|\geqslant r}|z|^\gamma\kappa(t,x,\mathrm{v},z)\nu(\mathrm{d}z)\leqslant c_1 2^{2-\gamma}r^{\gamma-\alpha}.$$

- Hence $\nu_s(\mathrm{d}z):=\kappa_0(s,z)\nu(\mathrm{d}z)$ is a Lévy measure.

- Let $N(\mathrm{d}t, \mathrm{dv})$ be the Poisson random measure on $\mathbb{R}^{d+1}$ with intensity measure $\nu_t(\mathrm{dv})\mathrm{d}t$, and $\tilde{N}(\mathrm{d}t, \mathrm{dv}) := N(\mathrm{d}t, \mathrm{dv}) - \nu_t(\mathrm{dv})\mathrm{d}t$ the compensated Poisson random measure.
- For $s \leqslant t$, define

$$L_{s,t} := \int_s^t \int_{|\mathrm{v}|\leqslant 1} \mathrm{v}\tilde{N}(\mathrm{d}r, \mathrm{dv}) + \int_s^t \int_{|\mathrm{v}|>1} \mathrm{v}N(\mathrm{d}r, \mathrm{dv}),$$

and $K_{s,t} = (X_{s,t}, V_{s,t}) := (\int_s^t L_{s,r}\mathrm{d}r, L_{s,t})$.
- Notice that if $X_{s,s} = x$ and $V_{s,s} = \mathrm{v}$, then

$$K^{x,\mathrm{v}}_{s,t} = (x + (t-s)\mathrm{v} + \int_s^t L_{s,r}\mathrm{d}r, \mathrm{v} + L_{s,t}).$$

- The infinitesimal generator of $K^{x,\mathrm{v}}_{s,t}$ is $\mathscr{L}^{(\alpha)}_{0,\mathrm{v}} + \mathrm{v}\cdot\nabla_x$, where

$$\mathscr{L}^{(\alpha)}_{0,\mathrm{v}} u(t,x,\mathrm{v}) := \mathrm{p.v.} \int_{\mathbb{R}^d} \Big(u(x, \mathrm{v}+z) - u(x,\mathrm{v})\Big)\kappa_0(t,z)\nu(\mathrm{d}z),$$

which means that

$$u(t,x,\mathrm{v}) = \mathbb{E}(\int_0^t f(s, K^{x,\mathrm{v}}_{s,t})\mathrm{d}s) + \mathbb{E}(\varphi(K^{x,v}_{0,t}))$$

is a classical solution of (1.3) when $\kappa(t,x,\mathrm{v},z) \equiv \kappa_0(t,z)$.

- By Itô formula, this classical solution is unique. Therefore, we obtain a presentation of the (1.3) when κ is independent with x, v.
- Next, we can get the existence and smoothness of the density of $K_{s,t}$ denoted by $p_{s,t}(x,\mathrm{v})$. Notice that

$$P_{s,t}g(x,\mathrm{v}) := \mathbb{E}(g(K_{s,t}^{x,\mathrm{v}})) = \int_{\mathbb{R}^{2d}} p_{s,t}(y,w)g(x+(t-s)\mathrm{v}+y,\mathrm{v}+w)\mathrm{d}y\mathrm{d}w.$$

- For simplify, we define $\Gamma_{s,t}g(x,\mathrm{v}) := g(x+(t-s)\mathrm{v},\mathrm{v})$ and then we have $P_{s,t}g = \Gamma_{s,t}p_{s,t} * g$.

Crucial lemma

Lemma 6

(i) *For any* $\gamma \geqslant 0$ *and* $\beta_i \in [0, \alpha)$ *for* $i = 1, 2$ *and* $n, m \in \mathbb{N}_0$*, operator* $\mathscr{H}_j$ *is* $\Delta_j^a \Gamma_{s,t}$ *or* $\Gamma_{s,t}^{-1} \Delta_j^a \Gamma_{s,t}$*, there is a constant* $C > 0$*(which is independent with* κ_0 *only depends on* c_1*) such that for any nonnegative measurable* $f : \mathbb{R}_+ \to \mathbb{R}_+$*,* $\theta \geqslant \beta_2$ *and all* $j \in \mathbb{N}_0$*,*

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^d} |x|^{\beta_1} |\mathrm{v}|^{\beta_2} |\nabla_{\mathrm{v}}^n \nabla_x^m \mathscr{H}_j p_{s,t}(x, \mathrm{v})| f(s) \mathrm{d}x \mathrm{d}\mathrm{v} \mathrm{d}s \\ &\leqslant C 2^{[((1+\alpha)m+n)-\gamma-((1+\alpha)\beta_1+\beta_2)]j} \int_0^t (t-s)^{-\gamma/\alpha} f(s) \mathrm{d}s. \end{aligned} \tag{3.2}$$

(ii) *For any* $T > 0$*,* $p, q \in [1, +\infty]$ *and* $\gamma \geqslant 0$*, there is a constant* C *such that for all* $j \in \mathbb{N}_0$ *and* $0 \leqslant s < t \leqslant T$

$$\|\Delta_j^a \Gamma_{s,t} p_{s,t}\|_{\mathbb{L}^{p,q}} \leqslant C 2^{-\gamma j} (t-s)^{-\frac{\gamma+\Theta}{\alpha}},$$

where

$$\Theta = (1+\alpha)(1 - \frac{d}{p}) + 1 - \frac{d}{q}.$$

The key point of proof

- Notice that

$$N(t,\mathrm{dv})-N(s,\mathrm{dv})\overset{(d)}{=}N(t-s,\mathrm{dv})\overset{(d)}{=}(t-s)^{\frac{1}{\alpha}}N_1(1,\mathrm{dv}),$$

where the intensity measure of N_1 is $\kappa((t-s),\mathrm{v})\nu(\mathrm{dv})$. Therefore

$$V_{s,t}\overset{(d)}{=}(t-s)^{\frac{1}{\alpha}}\tilde{V}_{0,1},$$

where the Lévy measure of $\tilde{V}_{0,r}$ is $\kappa((t-s)r,\mathrm{v})\nu(\mathrm{dv})$. Furthermore

$$K_{s,t}\overset{(d)}{=}((t-s)^{\frac{1+\alpha}{\alpha}}\tilde{X}_{0,1},(t-s)^{\frac{1}{\alpha}}\tilde{V}_{0,1}),$$

where $\tilde{X}_{0,1}=\int_0^1\tilde{V}_{0,r}\mathrm{d}r$.

- We denote by $\bar{p}_{0,1}$ the density of $(\tilde{X}_{0,1},\tilde{V}_{0,1})$, then

$$p_{s,t}(x,\mathrm{v})=(t-s)^{-(\frac{d}{\alpha}+\frac{(1+\alpha)d}{\alpha})}\bar{p}_{0,1}((t-s)^{-\frac{1+\alpha}{\alpha}}x,(t-s)^{-\frac{1}{\alpha}}\mathrm{v}).$$

The key point of proof

- Notice that
$$N(t,\mathrm{dv})-N(s,\mathrm{dv})\overset{(d)}{=}N(t-s,\mathrm{dv})\overset{(d)}{=}(t-s)^{\frac{1}{\alpha}}N_1(1,\mathrm{dv}),$$
where the intensity measure of N_1 is $\kappa((t-s),\mathrm{v})\nu(\mathrm{dv})$. Therefore
$$V_{s,t}\overset{(d)}{=}(t-s)^{\frac{1}{\alpha}}\tilde{V}_{0,1},$$
where the Lévy measure of $\tilde{V}_{0,r}$ is $\kappa((t-s)r,\mathrm{v})\nu(\mathrm{dv})$. Furthermore
$$K_{s,t}\overset{(d)}{=}((t-s)^{\frac{1+\alpha}{\alpha}}\tilde{X}_{0,1},(t-s)^{\frac{1}{\alpha}}\tilde{V}_{0,1}),$$
where $\tilde{X}_{0,1}=\int_0^1\tilde{V}_{0,r}\mathrm{d}r$.
- We denote by $\bar{p}_{0,1}$ the density of $(\tilde{X}_{0,1},\tilde{V}_{0,1})$, then
$$p_{s,t}(x,\mathrm{v})=(t-s)^{-(\frac{d}{\alpha}+\frac{(1+\alpha)d}{\alpha})}\bar{p}_{0,1}((t-s)^{-\frac{1+\alpha}{\alpha}}x,(t-s)^{-\frac{1}{\alpha}}\mathrm{v}).$$
- Condition
$$\inf_{\omega\in\mathbb{S}^{d-1}}\int_{|z|\leqslant r}(\omega\cdot z)^2\kappa(t,x,\mathrm{v},z)\nu(\mathrm{d}z)\geqslant c_0r^{2-\alpha},\tag{3.3}$$
guarantee that for any $n,m\in\mathbb{N}_0$ and $\beta_i\in[0,\alpha)$, there is a constant C such that
$$\int_{\mathbb{R}^{2d}}|x|^{\beta_1}|\mathrm{v}|^{\beta_2}|\nabla_x^n\nabla_{\mathrm{v}}^m\bar{p}_{0,1}(x,\mathrm{v})|\mathrm{d}x\mathrm{dv}\leqslant C.$$

Part 4: Our approach

Our approach

- Firstly, we use a technology of translate alone the characteristic line $(x_0 + t\mathrm{v}_0, \mathrm{v}_0)$, let $\tilde{u}(t,x,\mathrm{v}) := u(t, x + x_0 + t\mathrm{v}_0, \mathrm{v} + \mathrm{v}_0)$ and get a new equation:

$$\begin{cases} \partial_t \tilde{u}(t,x,\mathrm{v}) + \mathscr{L}^{(\alpha)}_{\tilde{\kappa},\mathrm{v}} \tilde{u}(t,x,\mathrm{v}) + \mathrm{v} \cdot \nabla_x \tilde{u}(t,x,\mathrm{v}) = \tilde{f}(t,x,\mathrm{v}), \\ \tilde{u}(0,x,\mathrm{v}) = \tilde{\varphi}(x,\mathrm{v}), \end{cases} \tag{4.1}$$

where $\tilde{f}(t,x,\mathrm{v}) = f(t, x + x_0 + t\mathrm{v}_0, \mathrm{v} + \mathrm{v}_0)$, $\tilde{\varphi}(x,\mathrm{v}) = \varphi(x + x_0, \mathrm{v} + \mathrm{v}_0)$ and $\tilde{\kappa}(x,\mathrm{v}) = \kappa(x + x_0 + t\mathrm{v}_0, \mathrm{v} + \mathrm{v}_0)$.

- Then we have the following presentation

$$\tilde{u}(t,x,\mathrm{v}) = \int_0^t P_{s,t}\left(\mathscr{L}^{(\alpha)}_{\tilde{\kappa}} - \mathscr{L}^{(\alpha)}_{\tilde{\kappa}_0}\right)\tilde{u}(s,x,\mathrm{v})\mathrm{d}s + \int_0^t P_{s,t}\tilde{f}(s,x,\mathrm{v})\mathrm{d}s + P_{0,t}\tilde{\varphi}(x),$$

where $\tilde{\kappa}_0(t) = \kappa(x_0 + t\mathrm{v}_0, \mathrm{v}_0)$ and $\mathscr{L}^{(\alpha)}_{\tilde{\kappa}_0}$ is a infinitesimal generator of some process introduced in the part 3.

- Next step is a highlight point. We operator the block operator Δ_j^a on both sides and only look at the point zero:

$$\Delta_j^a \tilde{u}(t,0,0) = \int_0^t \Delta_j^a P_{s,t}\Big(\mathscr{L}_{\tilde{\kappa}}^{\alpha} - \mathscr{L}_{\tilde{\kappa}_0}^{(\alpha)}\Big)\tilde{u}(s,0,0)\mathrm{d}s + \int_0^t \Delta_j^a P_{s,t}\tilde{f}(s,0,0)\mathrm{d}s \\ + \Delta_j^a P_{0,t}\tilde{\varphi}(0,0).$$

- Notice that $\Delta_j^a u(t, x_0 + t\mathrm{v}_0, \mathrm{v}_0) = \Delta_j^a \tilde{u}(t,0,0)$. We take the supremum of the initial point (x_0, v_0) and get the estimate of $\|\Delta_j^a u(t)\|_\infty$. Then by taking sipremum of j, we have for some $\vartheta > -1$ and any $\gamma \in [0,\alpha)$:

$$\|u(t)\|_{B^{\gamma}_{a,\infty}} \lesssim \int_0^t (t-s)^{\vartheta}\|u(s)\|_{B^{\gamma}_{a,\infty}}\mathrm{d}s + t^{-\frac{1}{\alpha}(\gamma+\Theta)}\|\phi\|_{\mathbb{L}^{p,q}}.$$

- Notice that a highlight point here is that we turn the convolution $P_{s,t}f$ into an inner product $\langle p_{s,t}, f\rangle$. Therefore, we use our crucial lemma and get the regularity of the space.

Volterra-type Gronwall inequality

Lemma 7 (Volterra-type Gronwall inequality)

Assume $A > 0$. *For any* $\theta, \vartheta > -1$ *and* $T > 0$, *there exists a constant* $C = C(A, \theta, \vartheta, T) \geqslant 0$ *such that if locally integrable functions* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfy*

$$f(t) \leqslant A \int_0^t (t-s)^\theta f(s) \mathrm{d}s + At^\vartheta, \quad t \in (0, T],$$

then

$$f(t) \leqslant Ct^\vartheta, \quad t \in (0, T].$$

- When $\gamma + \Theta < \alpha$, $t^{-\frac{1}{\alpha}(\gamma+\Theta)}$ is a local integral function on $[0, T]$. We obtain main result for $\gamma \in [0, \alpha)$ and $\Theta < \alpha - \gamma$.
- To lift the limitation of γ from $[0, \alpha)$ to $[0, \alpha + \beta)$, we use a lift lemma by the flow property.
- The proof can be found in [1].

[1] X. Zhang, Stochastic Volterra equations in Banach spaces and stochastic partial differential equation. *J.Funct. Anal., 258 (2010), 1361-1425.*

Lift lemma

Lemma 8

Assume $\alpha \in (0,2)$ *and* $f \equiv 0$. *Under condition* $(\mathbf{H}^{\kappa}_{\beta,\beta})$, *for any*

$$\gamma \in (\alpha, \alpha+\beta), \quad \delta \in [0,\alpha),$$

there is a constant C_T *such that for all* $\varphi \in C_0^\infty(\mathbb{R}^d)$, *all classical solutions of* (1.3) *and all* $t \in (0,T]$,

$$\|u(t)\|_{B^{\gamma}_{a,\infty}} \leqslant C_T t^{-\frac{\delta}{\alpha}} \|\varphi\|_{B^{\gamma-\delta}_{a,\infty}}. \tag{4.2}$$

Lemma 9

Assume $\alpha \in (0,2)$. *Under the condition* $(\mathbf{H}^{\kappa}_{\theta,\beta})$, *for any* $\gamma \in (\alpha-\beta,\alpha)$ *and* $\eta \in (0,\theta)$, *there is a constant* $C>0$ *such that for all* $0 < t \leqslant T$,

$$\|u(t)\|_{\mathbf{C}_x^{\frac{\eta}{1+\alpha}}(\mathbf{C}_a^{\gamma})} \leqslant C t^{-\frac{\gamma}{\alpha}} \|\phi\|_{\mathbf{C}_x^{\frac{\eta}{1+\alpha}}}.$$

- Notice that $u_s(t,x,\mathrm{v}) := u(t+s,x,\mathrm{v})$ is the classical solution of (1.3) when $\varphi = u(s)$. Then by the uniqueness of the classical solution, we have

$$\|u(t)\|_{\mathbf{B}^{\gamma}_{a,\infty}} \lesssim t^{\frac{\delta}{\alpha}} \|u(\frac{t}{2})\|_{\mathbf{B}^{\gamma-\delta}_{a,\infty}}.$$

Existence

- For the existence of the classical solution, we use the probability representation.
- Firstly, we let $\kappa_\varepsilon := (\kappa + \varepsilon) \wedge \frac{1}{\varepsilon}$ and κ_ε has both lower bounded and upper bounded. We make a modifier of it and assume $\kappa_\varepsilon \in C_b^\infty$.
- When $\nu(\mathrm{d}z) = |z|^{-d-\alpha}\mathrm{d}z$, there is an interesting transform lemma:

Lemma 10 (H.-Peng-Zhang 2019)

Given $d \in \mathbb{N}$ and $c_0 > 1$, let $\kappa(x,z) : \mathbb{R}^d \times \mathbb{R}^d \to [c_0^{-1}, c_0]$ be a smooth function with bounded derivatives. For any $\alpha \in (0,2)$, there is a measurable map $\sigma(x,z) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ such that for any nonnegative measurable function f,

$$\int_{\mathbb{R}^d} f \circ \sigma(x,z) \frac{\mathrm{d}z}{|z|^{d+\alpha}} = \int_{\mathbb{R}^d} f(z)\kappa(x,z)\frac{\mathrm{d}z}{|z|^{d+\alpha}}.$$

Moreover, σ enjoys the following properties:

- $\sigma(x,0) = 0$ *and if* $\kappa(x,-z) = \kappa(x,z)$, *then* $\sigma(x,-z) = -\sigma(x,z)$.
- *For all $i, j \in \mathbb{N}_0$, there is a $C_{ij} > 0$ such that for all $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^d$,*

$$|\nabla_x^i \nabla_z^j \sigma(x,z)| \leqslant C_{ij}|z|^{1-j},$$

where C_{ij} is a polynomial of $\|\nabla_x^m \nabla_z^n \kappa\|_\infty$, $m = 1, \cdots, i, n = 0, \cdots, j$.

- When $\kappa_\varepsilon \in C_b^\infty$, we find a σ_ε satisfy the condition in transform lemma. Let $(X_{s,t}, V_{s,t})$ be the solution of following SDE

$$\begin{cases} \mathrm{d}X^\varepsilon_{s,t} = V_{s,t}\mathrm{d}L_t, \\ \mathrm{d}V^\varepsilon_{s,t} = \int_{\mathbb{R}^d} \sigma_\varepsilon(X^\varepsilon_{s,t-}, V^\varepsilon_{s,t-}, z)N(\mathrm{d}z, \mathrm{d}t), \end{cases}$$

  where L_t is a standard α-stable process and $N(t, A) := \sum_{s\leqslant t} \mathbf{1}_{\Delta L_s \in A}$.
- Notice that this time

$$u^\varepsilon(t, x, \mathrm{v}) := \mathbb{E}(\int_0^t f(s, K^\varepsilon_{s,t}(x, \mathrm{v}))\mathrm{d}s) + \mathbb{E}(\varphi(K^\varepsilon_{s,t}(x, \mathrm{v})))$$

  where $K^\varepsilon_{s,t}(x, \mathrm{v}) := (X^\varepsilon_{s,t} + x + (t-s)\mathrm{v}, V^\varepsilon_{s,t} + \mathrm{v})$, is a classical solution of (1.3) when φ and f is smooth.
- Finally, by Hölder estimate (Theorem 4), we obtain that $u^\varepsilon \to u$ which is a classical solution of (1.3).

Part 5: Some techniques

- In our work, $\Delta_j^a \Gamma_{s,t} \neq \Gamma_{s,t} \Delta_j^a$. This is very bad.
- Fortunately, we have the following observation

Lemma 11

For $t \geqslant 0$ and $j \in \mathbb{N}_0$, define

$$\Theta_j^t := \left\{ \ell \in \mathbb{N}_0 : 2^\ell \leqslant 2^4(2^j + t2^{(1+\alpha)j}),\ 2^j \leqslant 2^4(2^\ell + t2^{(1+\alpha)\ell}) \right\}.$$

(i) *Let $0 \leqslant s < t$ and $j \in \mathbb{N}$. For any $\ell \notin \Theta_j^{t-s}$, it holds that*

$$\langle \Delta_j^a f, \Gamma_{s,t} \Delta_\ell^a g \rangle = \int_{\mathbb{R}^{2d}} \Delta_j^a f(x, \mathrm{v}) \cdot \Gamma_{s,t} \Delta_\ell^a g(x, \mathrm{v}) \mathrm{d}x \mathrm{d}\mathrm{v} = 0. \tag{5.1}$$

(ii) *For any $\beta > 0$, there is a constant $C = C(c_1, \beta) > 0$ such that for all $j \in \mathbb{N}$ and $t \geqslant 0$,*

$$\sum_{\ell \in \Theta_j^t} 2^{-\beta\ell} \leqslant C\left(2^{-j} + t2^{(\alpha-1)j}\right)^\beta, \quad \sum_{\ell \in \Theta_j^t} 2^{\beta\ell} \leqslant C\left(2^j + t2^{(1+\alpha)j}\right)^\beta. \tag{5.2}$$

- This lemma tell us that

$$\Delta_j^a \Gamma_{s,t} = \sum_{\ell=0}^{\infty} \Delta_j^a \Gamma_{s,t} \Delta_\ell^a \approx \Delta_j^a \Gamma_{s,t} \Delta_j^a.$$

- The proof can be found in [1].

[1] Z. Hao, M. Wu, and X. Zhang, Schauder's estimate for nonlocal kinetic equations and its applications. Available at arXiv:1903.09967.

- There is a useful commutator estimate.

Lemma 12

Assume $\alpha \in (0,2)$ *and condition* $(\mathbf{H}^{\kappa}_{\beta})$ *with* $\beta \in (0, \alpha \wedge 1)$. *Define*

$$f_z(x, \mathrm{v}) = f(x, v+z) - f(x,v) - z^{\alpha}\nabla_v f(x,v)$$

where $z^{\alpha} := z\mathbf{1}_{\alpha>1} + z\mathbf{1}_{|z|<1}\mathbf{1}_{\alpha=1}$.

(i) *For any* $\varepsilon > 0$, *there is a constant* C *such that for all* $f \in C^{\infty}_b$, $j \in \mathbb{N}_0$ *and* $x, \mathrm{v} \in \mathbb{R}^d$

$$\int_{|z|\leqslant 1} |[\Delta^a_j, \kappa] f_z(x, \mathrm{v})| \nu(\mathrm{d}z) \leqslant 2^{-j\beta} C \|f\|_{C^{\alpha+\varepsilon}_{\mathrm{v}}}. \tag{5.3}$$

(ii) *For any* $\eta \in (-\beta, 0]$ *and* $\varepsilon > 0$, *there is a constant* C *such that for all* $f \in C^{\infty}_b$, $j \in \mathbb{N}$ *and* $x, \mathrm{v} \in \mathbb{R}^d$

$$\int_{|z|\leqslant 1} |[\Delta^a_j, \kappa] f_z(x, \mathrm{v})| \nu(\mathrm{d}z) \leqslant 2^{-j(\beta+\eta)} C \|f\|_{C^{\alpha+\varepsilon+\eta}_{\mathrm{v}}}.$$

- The key to the proof of this lemma is the Bony decomposition.

$$fg = \sum_{i,j=0}^{\infty} \Delta_i f \Delta_j g = \sum_{i>j+1} \Delta_i f \Delta_j g + \sum_{j>i+1} \Delta_i f \Delta_j g + \sum_{(i-j|\leqslant 1} \Delta_i f \Delta_j g$$
$$:= f > g + f < g + f \circ g.$$

Part 6: Future works

Future works

- We prove the existence of the fundamental solution of the non-local kinetic equation (1.3). Furthermore, we show that this is a $\mathbb{L}^{p,q}$ solution. However this result does not imply that it is continuous and satisfies

$$\partial_t p^{\kappa}_{s,t} + \mathscr{L}^{(\alpha)}_{\kappa,\mathrm{v}} p^{\kappa}_{s,t} + \mathrm{v} \cdot \nabla_x p^{\kappa}_{s,t} = 0.$$

- We can not deal with the existence when $\nu(\mathrm{d}z) \neq |z|^{-d-\alpha}\mathrm{d}z$. The reason is there is not a transform lemma when Lévy measure is singular. We try to use some continuity methods and vanishing viscosity approach, but they are not work.
- Actually, the density f in Boltzmann equation is a distribution. It may not have some high regularity. However, in our model, we consider the classical solution with $\mathbf{C}^{1\vee\alpha+\varepsilon}_x \cap \mathbf{C}^{\alpha+\varepsilon}_{\mathrm{v}}$ regularity. We want to built a mild or weak solution theorem in the next step.

Thanks for your attention!