Supercritical SDEs driven by fractional Brownian motion with divergence free drifts

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Regularization by noise

- Consider the following SDE:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t,$$

where $(W_t)_{t\geq 0}$ is a standard d-dimensional Brownian motion and $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a measurable function.

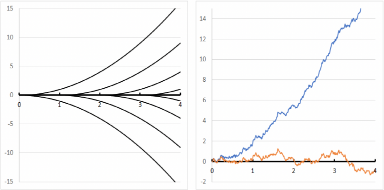

Figure 1: Regularization by noise.
LHS: infinity solutions to the ODE $\mathrm{d}X_t = \mathrm{sign}(X_t)\sqrt{|X_t|}\mathrm{d}t$;
RHS: the unique solution (blue) to SDE $\mathrm{d}X_t = \mathrm{sign}(X_t)\sqrt{|X_t|}\mathrm{d}t + \mathrm{d}W_t$ driven by the Brownian motion (yellow).

Scale analysis

- Suppose for some $p, q \in [1, \infty]$ and $\alpha \in \mathbb{R}$
$$b \in L_t^q \mathbf{H}_x^{\alpha,p} := L^q(\mathbb{R}_+; \mathbf{H}^{\alpha,p}(\mathbb{R}^d)),$$
and SDE admits a solution denoted by X. For $\lambda > 0$, we define
$$X_t^\lambda := \lambda^{-1} X_{\lambda^2 t}, \quad W_t^\lambda := \lambda^{-1} W_{\lambda^2 t}, \quad b^\lambda(t,x) := \lambda b(\lambda^2 t, \lambda x).$$
- Then we have
$$\mathrm{d}X_t^\lambda = b^\lambda(t, X_t^\lambda)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t^\lambda,$$
where
$$\|b^\lambda\|_{L^q(\mathbb{R}_+;\dot{\mathbf{H}}^{\alpha,p})} = \lambda^{1+\alpha-\frac{d}{p}-\frac{2}{q}} \|b\|_{L^q(\mathbb{R}_+;\dot{\mathbf{H}}^{\alpha,p})}.$$
- As $\lambda \to 0$,

Subcritical: $\frac{d}{p} + \frac{2}{q} < 1 + \alpha$;

Critical: $\frac{d}{p} + \frac{2}{q} = 1 + \alpha$;

Supercritical: $\frac{d}{p} + \frac{2}{q} > 1 + \alpha$.

Well-known results

SEU: Strong existence-uniqueness; WEU: Weak existence-uniqueness;
WE: Weak existence; EUE: Existence-uniqueness of energy solution.

Value of α	Subcritical	Critical	Supercritical
$\alpha = 0$	SEU: $V^{79}_{[1]}$, $KR^{05}_{[2]}$, $Z^{05,10}_{[3,4]}$	WEU&SEU: $BFGM^{19}_{[5]}$, $K^{21}_{[6]}$, $RZ^{21}_{[7,8]}$, $KM^{23}_{[9]}$	WE: $ZZ^{21}_{[10]}$
$\alpha \in [-\frac{1}{2}, 0)$	WEU: $BC^{01}_{[11]}$, $FIR^{17}_{[12]}$, $ZZ^{17}_{[13]}$	–	–
$\alpha \in [-1, -\frac{1}{2})$	EUE: $GP^{23}_{[14]}$	–	WEU: $HZ^{23}_{[15]}$, EUE: $GP^{24}_{[16]}$, $G^{24}_{[17]}$

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SDE driven by fBM

- d-dimensional Fractional Brownian motion (fBM) $W_t^H = (W_t^{H,1}, ..., W_t^{H,d})$: for $H \in (0,1)$,

$$\mathbb{E}(W_t^{H,i} W_s^{H,j}) = (t^{2H} + s^{2H} - |t-s|^{2H}) 1_{i=j}.$$

For $H \in (1, \infty) \backslash \mathbb{N}$, $W_t^H := \int_0^t W_s^{H-1} \mathrm{d}s$.

- It is not a Markov process.
- Consider the SDE driven by fBM:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}W_t^H, \tag{1}$$

where $b \in L_t^q \mathbf{H}_x^{\alpha,p}$.

- Scaling and critical conditions: $W_t^H \overset{(d)}{=} \lambda^{-H} W_{\lambda t}^H$.

Subcritical: $H(\frac{d}{p} - \alpha) + \frac{1}{q} < 1 - H$;

Critical: $H(\frac{d}{p} - \alpha) + \frac{1}{q} = 1 - H$;

Supercritical: $H(\frac{d}{p} - \alpha) + \frac{1}{q} > 1 - H$.

Well-known results

- Subcritical:

  Nualart-Ouknine 02: $d = 1$. (SEU) $H \in (0, \frac{1}{2}]$, $\alpha = 0$, $p, q \geq 2$ and $\frac{H}{p} + \frac{1}{q} < \frac{1}{2}$;

  $$H \in (\tfrac{1}{2}, 1), b \in C_x^{1-\frac{1}{2H}+} \cap C_t^{H-\frac{1}{2}+}.$$

  H.-Röckner-Zhang 24: (WEU) $H \in (0, \frac{1}{2}]$, $\alpha = 0$, $p, q \geq \frac{1}{1-H}$ and $\frac{Hd}{p} + \frac{1}{q} < \frac{1}{2}$.

  Lê 20: (SEU) $\alpha = 0$, $\frac{Hd}{p} + \frac{1}{q} < \frac{1}{2} - H$.

  Butkovsky-Lê-Mytnik 23: $b = b(x)$. (WE) $\alpha = 0$ and $\frac{Hd}{p} < 1 - H$.

  (SEU) $\alpha = 0$ and $H^2 + H(1 + 2/p) < 1$.

  Gatellier-Gubinelli 16: (path-by-path well-posedness) $H \in (0, 1)$, $b = b(x) \in C^{1-\frac{1}{2H}+}$.
  Gerencsér 23: $H > 1$.

  Butkovsky-Mytnik 24: (WEU) $H \in (0, \frac{1}{2})$, $b = b(x) \in C^{\frac{1}{2}-\frac{1}{2H}+}$.

  Galeati-Gerencsér 25: (path-by-path well-posedness)

  $$q \leq 2 \text{ and } H(\tfrac{d}{p} - \alpha) + \tfrac{1}{q} < 1 - H.$$

  H.-Röckner-Zhang 24: (WEU) $H \in (0, \frac{1}{2})$, $\alpha = 0$, $p, q \geq \frac{1}{1-H}$, and

  $$\frac{Hd}{p} + \frac{1-H}{q} < (1-H)^2.$$

- Supercritical:

  Butkovsky-Gallay 23: (WE) $\alpha = 0$ and $\frac{Hd}{p} + \frac{1-H}{q} < 1 - H$.

Well-known results

- DDSDE and Euler-Murayama approximation: Galeati-Harang-Mayorcas 22, 23, Butkovsky-Dareiotis-Gerencsér 21, ...

Our aim

BM Case ($H = \frac{1}{2}$):

- From Subcritical, Critical: to Supercritical: additional conditions on div$b$.
- PDEs help us to understand and solve SDEs (Itô's formula).

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BM Case ($H = \frac{1}{2}$):

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- PDEs help us to understand and solve SDEs (Itô's formula).

FBM Cases ($H \neq \frac{1}{2}$):

- Our aim:
  supercritical case with some conditions on div$b$.
- Difficulty:
  Non-Markovian solutions. No PDE techniques.

Weak existence

($\mathbf{H}_b$) We assume that

$$(\mathrm{div} b)_- \in L_t^1 L_x^\infty \quad \text{and} \quad \frac{b(t,x)}{1+|x|} \in L_t^1(L_x^1 + L_x^\infty).$$

Theorem 1 (H.-Galeati 2025+)

*Let $H \in (0,\infty)\backslash\mathbb{N}$ and assume **($\mathbf{H}_b$)** holds. Then for Lebesgue almost every $x \in \mathbb{R}^d$, there exists at least one weak solution to SDE (1) with $X_0 = x$.*
Moreover, whenever $\mathcal{L}(X_0) \in L_x^1$, there exists at least one weak solution to SDE (1).

Weak uniqueness

Theorem 2 (H.-Galeati 2025+)

Assume $(\boldsymbol{H}_b)$ *and one of the following conditions hold:*

(i) $H \in (0,1/2]$ *and* $b \in L_t^{\frac{1}{1-H}} L_{loc}^{\frac{1}{1-H}}$;

(ii) $H \in (1/2,1)$ *and* $b = b(x) \in \mathbf{H}_{loc}^{1-\frac{1}{2H}+,2}(\mathbb{R}^d)$.

Then for Lebesgue almost every $x \in \mathbb{R}^d$*, there exists a unique weak solution to SDE* (1) *with* $X_0 = x$ *satisfying*

(i) $\int_0^T |b(s,X_s)|^{\frac{1}{1-H}} \mathrm{d}s < \infty, \quad \mathbb{P}-a.s.$*, when* $H \in (0,1/2]$;

(ii) $\|b(X_\cdot)\|_{\mathbf{H}^{\beta,2}([0,T])} < \infty, \quad \mathbb{P}-a.s.$ *with some* $\beta > H - 1/2$ *when* $H \in (1/2,1)$.

- For any b satisfying the above condition, we can associate a Lebesgue-a.e. uniquely defined family of probability measure $\{\mathbf{P}^x(b)\}_{x\in\mathbb{R}^d} \subset \mathscr{P}(\mathbb{C}_T)$ by setting $\mathbf{P}^x(b) = \mathscr{L}(X^x)$, where X^x is the unique-in-law solution to the SDE (1) starting from x.
- For any $\mathscr{L}(X_0) \in L^1(\mathbb{R}^d)$, $\mathscr{L}(X) = \int_{\mathbb{R}^d} \mathbf{P}^x(b)\mathscr{L}(X_0)(x)\mathrm{d}x$.

Continuity of $b \to \mathbf{P}^x(b)$

Theorem 3 (H.-Galeati 2025+)

Under the same condition in weak uniqueness theorem, let $\{b^n\}_{n\in\mathbb{N}}$ be a sequence of functions such that

$$\sup_{n\in\mathbb{N}} \left\| \frac{b^n}{1+|\cdot|} \right\|_{L_t^1(L_x^1+L_x^\infty)} + \sup_{n\in\mathbb{N}} \|(\mathrm{div} b^n)_-\|_{L_t^1 L_x^\infty} < \infty,$$

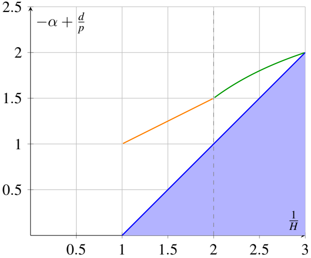
and

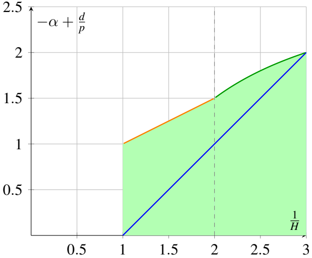
$$\begin{cases} b^n \to b \text{ in } L_t^{\frac{1}{1-H}} L_{loc}^{\frac{1}{1-H}},\ H \in (0, \frac{1}{2}]; \\ b^n = b^n(x), \quad b^n \to b \text{ in } \mathbf{H}_{loc}^{1-\frac{1}{2H}+,1},\ H \in (\frac{1}{2}, 1). \end{cases}$$

Then

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \|\mathbf{P}^x(b_n) - \mathbf{P}^x(b)\|_{\mathrm{var}} (1+|x|)^{-d-1} \mathrm{d}x = 0.$$

- When $d = 3$ and $b \in \mathbf{H}^{\alpha,p}(\mathbb{R}^d)$ with $\alpha \geq (1 - \frac{1}{2H})1_{H>1/2}$:
  - Subcritical condition (in blue) and our conditions (in green)





— $-\alpha H + \frac{dH}{p} = 1 - H$

— $p = \frac{1}{1-H}, \; H \in (0, \frac{1}{2}]$

— $\alpha = 1 - \frac{1}{2H}, p = 2, \; H \in (\frac{1}{2}, 1)$

Examples

- Ornstein-Uhlenbeck process.
  We note that $b(t,x) = \kappa x + \tilde{b}(t,x)$ satisfies the conditions with any $\kappa \in \mathbb{R}$ and divergence-free $\tilde{b} \in L_t^{\frac{1}{1-H}}(L_x^{\frac{1}{1-H}} + L_x^\infty)$ when $H \in (0, \frac{1}{2}]$.
  When $H \in (\frac{1}{2}, 1)$, $b(x) = \kappa x + \tilde{b}(x)$ satisfies the conditions with any $\kappa \in \mathbb{R}$ and divergence-free $\tilde{b} \in H^{1-\frac{1}{2H}+} + C^{1-\frac{1}{2H}+}$.

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- Stochastic Lagrangian paths for the Navier-Stokes equation.
  Let u be a Leray solution to 3D Navier-Stokes equations. Then $u \in L_t^{\infty} L_x^2$ and $\mathrm{div} u = 0$ which satisfies the conditions.

Examples

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  When $H \in (\frac{1}{2}, 1)$, $b(x) = \kappa x + \tilde{b}(x)$ satisfies the conditions with any $\kappa \in \mathbb{R}$ and divergence-free $\tilde{b} \in H^{1-\frac{1}{2H}+} + C^{1-\frac{1}{2H}+}$.

- Stochastic Lagrangian paths for the Navier-Stokes equation.
  Let u be a Leray solution to 3D Navier-Stokes equations. Then $u \in L_t^\infty L_x^2$ and $\mathrm{div} u = 0$ which satisfies the conditions.

- Letting $d = 3$ and for $\alpha \in (0, d+1)$, we consider the singular potential

$$U(x) = |x|^{-\alpha+1}, \text{ and } \nabla^\perp := (\partial_{x_3} - \partial_{x_2}, -\partial_{x_1} - \partial_{x_3}, \partial_{x_2} - \partial_{x_1}).$$

  Then we have

$$\nabla^\perp U \in L^{\frac{1}{1-H}} + L^\infty, \quad \text{if } H \in (0, \tfrac{1}{2}] \text{ and } \alpha \in (0, 3-3H),$$

  which is in the supercritical regime when $H \in (\frac{1}{3}, \frac{1}{2}]$.
  When $H \in (\frac{1}{2}, 1)$, we have

$$\nabla^\perp U \in H^{1-\frac{1}{2H}+} + C^{1-\frac{1}{2H}+}, \quad \text{if } \alpha \in (0, \tfrac{3H+1}{2H} - 1).$$

Quantitative stability

Case $H \in (0, \frac{1}{2}]$:

Theorem 4 (H.-Galeati 2025+)

Let $H \in (0, 1/2]$. Assume that X_0 admits a density $\rho_0 \in L^2(\mathbb{R}^d)$. Let b^1, b^2 be drifts satisfying

$$b^i \in L_t^{\frac{1}{1-H}}(L_x^2 + L_x^\infty), \quad \|(\mathrm{div} b^i)_-\|_{L_t^1 L_x^\infty} < \infty \quad \textit{for } i = 1, 2.$$

Then there exists a constant $C = C(T, H, p_0, p, \|(\mathrm{div} b^1)_-\|_{L_t^1 L_x^\infty}) > 0$ such that

$$\|\mathbf{P}^1 - \mathbf{P}^2\|_{\mathrm{var}} \leq C(1 + \|\rho_0\|_{L_x^2}) \|b^1 - b^2\|_{L_t^{\frac{1}{1-H}}(L_x^2 + L_x^\infty)}$$

where $\mathbf{P}^i$ denote the law of the unique weak solutions to SDE (1) with initial law $\mathcal{L}(X_0) = \rho_0(x)\mathrm{d}x$ and drift b^i, for $i = 1, 2$.

Quantitative stability

Case $H \in \left(\frac{1}{2}, 1\right)$:

Theorem 5 (H.-Galeati 2025+)

Let $H \in (1/2, 1)$. Assume that X_0 admits a density $\rho_0 \in L^2(\mathbb{R}^d)$. Let $b^1, b^2 : \mathbb{R}^d \to \mathbb{R}^d$ be drifts satisfying

$$b^i \in \mathbf{H}^{\alpha,p}(\mathbb{R}^d), \quad \|(\mathrm{div} b^i)_-\|_{L^\infty} < \infty \quad \textit{for } i = 1, 2.$$

with some $\alpha > 1 - \frac{1}{2H}$ and $p > 4H$.
Then there exists a constant $C > 0$ such that

$$\|\mathbf{P}^1 - \mathbf{P}^2\|_{\mathrm{var}} \leq C(1 + \|\rho_0\|_{L^2_x}) \|b^1 - b^2\|_{\mathbf{H}^{\alpha,p}}$$

where $\mathbf{P}^i$ denote the law of the unique weak solutions to SDE (1) with initial law $\mathcal{L}(X_0) = \rho_0(x)\mathrm{d}x$ and drift b^i, for $i = 1, 2$.

McKean-Vlasov SDE

Case $H \in (0, \frac{1}{2}]$:

Theorem 6 (H.-Galeati 2025+)

Let $H \in (0, 1/2]$. Assume that $(\mathrm{div} b)_- \in L_t^1 L_x^\infty$ and $b \in L_t^q (L_x^2 + L_x^\infty)$ with $q > \frac{1}{1-H}$. Then for any $\rho_0 \in L^q(\mathbb{R}^d)$, strong well-posedness holds for the following McKean-Vlasov SDE:

$$X_t = X_0 + \int_0^t (b_s * \mu_{X_s})(X_s) \mathrm{d}s + W_t^H, \tag{2}$$

where $\mu_{X_s} = \mathcal{L}(X_s)$ and $\mathcal{L}(X_0)$ admits the density ρ_0.

McKean-Vlasov SDE

Case $H \in (\frac{1}{2}, 1)$:

Theorem 7 (H.-Galeati 2025+)

Let $H \in (1/2, 1)$. Assume that $b = b(x)$ with $(\mathrm{div} b)_- \in L^\infty_x$ and $b \in \mathbf{H}^{\alpha,p}$ with some $\alpha > 1 - \frac{1}{2H}$ and $p > 4H$. Then for any $\rho_0 \in L^2(\mathbb{R}^d)$, strong well-posedness holds for the following McKean-Vlasov SDE:

$$X_t = X_0 + \int_0^t (b * \mu_{X_s})(X_s)\mathrm{d}s + W_t^H, \tag{3}$$

where $\mu_{X_s} = \mathcal{L}(X_s)$ and $\mathcal{L}(X_0)$ admits the density ρ_0.

Conservation flow

- Assume $(\mathrm{div} b)_- \in L^1_t L^\infty_x$ and consider the following ODE:

$$X^x_t = x + \int_0^t b(s, X^x_s)\mathrm{d}s.$$

- Majda-Bertozzi 02, *Vorticity and incompressible flow*

$$\int_{\mathbb{R}^d} |f(X^x_t)|\mathrm{d}x \leqslant \mathrm{e}^{\int_0^t \|(\mathrm{div} b(s))_-\|_{L^\infty}\mathrm{d}s} \|f\|_{L^1(\mathbb{R}^d)}.$$

Conservation flow

- Assume $(\mathrm{div} b)_- \in L_t^1 L_x^\infty$ and consider the following SDE:

$$X_t^x = x + \int_0^t b(s, X_s^x) \mathrm{d}s + W_t^H.$$

- Conservation (Zhang 13, RMI):

$$\int_{\mathbb{R}^d} |f(X_t^x)| \mathrm{d}x \leqslant \mathrm{e}^{\int_0^t \|(\mathrm{div} b(s))_-\|_{L^\infty} \mathrm{d}s} \|f\|_{L^1(\mathbb{R}^d)}.$$

Proof of existence

- Let $b_n(t,x) := (b(t,\cdot) * \rho_n)(x)$ and consider the approximate SDE:

$$X_t(n) = X_0 + \int_0^t b_n(s, X_s(n))\mathrm{d}s + W_t^H.$$

We also let $X_t^x(n)$ be the solution with $X_0^x(n) = x \in \mathbb{R}^d$.

Lemma 8

Let $H \in (0,\infty)\backslash\mathbb{N}$ and assume **($\mathbf{H}_b$)** *holds. Then for any $\rho_0 \in L^1(\mathbb{R}^d)$, $n \in \mathbb{N}$, $\lambda > 0$, $M > 1$, and $f \in L^1(\mathbb{R}_+ \times \mathbb{R}^d)$,*

$$\mathbb{P}\left(\int_0^T |f(s, X_s^n)|\mathrm{d}s > \lambda\right) \leq o_M(1) + \frac{Me^{T\|(\mathrm{div} b)_-\|_{L_t^1 L_x^\infty}}}{\lambda}\|f\|_{L^1(\mathbb{R}_+\times\mathbb{R}^d)}, \qquad (4)$$

with

$$o_M(1) := \int_{\mathbb{R}^d} 1_{|\rho(x)|>M}\rho_0(x)\mathrm{d}x.$$

Proof of existence

- Step 1: Tightness and convergence.
  - Fact 1: Compact embedding $BV(0,T) \hookrightarrow L^1([0,T])$;
  - Fact 2:

$$\begin{aligned}&\sup_n \mathbb{P}(\|X(n) - W^H\|_{\mathrm{BV}} > \lambda)\\ &\leq \sup_n \mathbb{P}\left(\int_0^T |b_n(s, X_s(n))| \mathrm{d}s > \lambda\right) \to 0, \quad \text{as } \lambda \to \infty.\end{aligned}$$

  - Conclusion:

$$X^n \to X \quad \text{in } L^1([0,T]).$$

- Step 2: (4) yields that

$$\lim_{n\to\infty} \int_0^T |b_n(s, X_s(n)) - b(s, X_s)| \mathrm{d}s = 0.$$

- Step 3: Based on the step 2, we have

$$X^n \to X \quad \text{in } C([0,T]),$$

which implies that X is a weak solution.

Proof of L^1-estimate (4)

- For $f \in L^1_{t,x}$, it follows from disintegration that for any $M > 1$

$$\mathbb{P}\left(\int_0^T |f(t, X_t(n))|\mathrm{d}t > \lambda\right) \le \int_{\mathbb{R}^d} \mathbb{P}\left(\int_0^T |f(t, X_t^x(n))|\mathrm{d}t > \lambda\right) \rho_0(x)\mathrm{d}x$$

$$\le \int_{|\rho_0(x)|>M} \rho_0(x)\mathrm{d}x + M \int_{|\rho_0(x)|\le M} \mathbb{P}\left(\int_0^T |f(t, X_t^x(n))|\mathrm{d}t > \lambda\right) \mathrm{d}x$$

$$\le o_M(1) + \frac{M}{\lambda} \int_{\mathbb{R}^d} \mathbb{E}\left(\int_0^T |f(t, X_t^x(n))|\mathrm{d}t\right) \mathrm{d}x \le o_M(1) + \frac{Me^{\|(\mathrm{div} b)_-\|_{L^1_t L^\infty_x}}}{\lambda} \|f\|_{L^1_{t,x}},$$

where we used the conservation property in the last inequality.

Proof of uniqueness: Girsanov's theorem

- Let W^H be a fBM in $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in[0,T]})$. It is well-known that there is a standard BM W such that

$$W_t^H = \int_0^t K_H(t,s)\mathrm{d}W_s.$$

where K_H is an explicit square-integrable Volterra kernel (see [2, Corollary 3.1]).

Theorem 9 (Girsanov's theorem, [2, Theorem 4.9])

For any $\mathscr{F}_t$-adapted process h with $\|h\|_{CM_H} := \|K_H^{-1}h\|_{L^2(0,T)} < \infty$, we define

$$Z_T := \exp\left(-\int_0^T (K_H^{-1}h)(s)\mathrm{d}W_s - \frac{1}{2}\|h\|_{CM_H}^2\right).$$

If $\mathbb{E}[Z_T] = 1$, then the measure $\mathbb{Q}$ given by $\mathrm{d}\mathbb{Q} := Z_T\mathrm{d}\mathbb{P}$ is again a probability measure and

$$\mathcal{L}_{\mathbb{Q}}(W^H + h) = \mathcal{L}_{\mathbb{P}}(W^H).$$

[2] L. Decreusefond and A. S. Üstünel, *Potential Anal.*, (1999).

Proof of uniqueness: Girsanov's theorem

- As a result of the Girsanov's theorem, the following uniqueness holds.

Lemma 10

Let $H \in (0,1)$. For any $\nu \in \mathscr{P}(\mathbb{R}^d)$, there is at most one weak solution to SDE (1) *with $\mathcal{L}(X_0) = \nu$ such that*

$$\left\| \int_0^{\cdot} b(s, w_s) \mathrm{d}s \right\|_{CM_H} < \infty, \quad \mathbf{P} - a.s. \tag{5}$$

- For $H \in (0, 1/2]$,

$$\left\| \int_0^{\cdot} f(s) \mathrm{d}s \right\|_{CM_H} \lesssim \|f\|_{L^{\frac{1}{1-H}}([0,T])}.$$

.

- For $H \in (1/2, 1)$ and any $\beta > H - 1/2$,

$$\left\| \int_0^{\cdot} f(s) \mathrm{d}s \right\|_{CM_H} \lesssim \|f\|_{\mathbf{H}^{\beta,2}([0,T])}.$$

Proof of uniqueness: Girsanov's theorem

- When $H \in (0, \frac{1}{2}]$, for $b \in L_t^{\frac{1}{1-H}} L_{loc,x}^{\frac{1}{1-H}}$, (4) gives us that

$$\|b(X_\cdot)\|_{L^{\frac{1}{1-H}}([0,T])} < \infty, \quad a.e.$$

- When $H \in (\frac{1}{2}, 1)$, for $b = b(x) \in \mathbf{H}_{loc,x}^{1-\frac{1}{2H}+,2}$, there is a $g \in L_{loc,x}^2$ such that

$$|b(x) - b(y)| \leq |x - y|^{1-\frac{1}{2H}+}(g(x) + g(y)).$$

Then we have

$$\|b(X_\cdot)\|_{\mathbf{H}^{\beta,2}([0,T])} \lesssim \|b(X_\cdot)\|_{L^2([0,T])} + \|g(X_\cdot)\|_{L^2([0,T])} + \|g(X_\cdot)\|_{L^2([0,T])}^{\frac{1}{1-\alpha}},$$

which implies that

$$\|b(X_\cdot)\|_{\mathbf{H}^{\beta,2}([0,T])} < \infty, \quad a.e.$$

Proof of stability

- Relative entropy:

$$H(\mathbf{P}_1 \mid \mathbf{P}_2) := \begin{cases} \mathbb{E}^{\mathbf{P}_1} \ln \left(\frac{\mathrm{d}\mathbf{P}_1}{\mathrm{d}\mathbf{P}_2} \right), & \text{if } \mathbf{P}_1 \ll \mathbf{P}_2, \\ \infty, & \text{otherwise.} \end{cases}$$

- ([3]) For any b_1 and b_2 satisfying our assumptions and smooth,

$$H(\mathbf{P}(b_1) \mid \mathbf{P}(b_2)) \leq \frac{1}{2}\mathbb{E}\| \int_0^{\cdot} (b_1(s, X_s(1)) - b_2(s, X_s(1))\mathrm{d}s\|^2_{CM_H}$$

- By Pinsker's inequality and conservation property,

$$\|\mathbf{P}(b_1) - \mathbf{P}(b_2)\|_{\mathrm{var}} \lesssim \begin{cases} \|b_1 - b_2\|_{L_t^{\frac{1}{1-H}} L_x^2}, & H \in (0, \frac{1}{2}]; \\ \|b_1 - b_2\|_{\mathbf{H}^{1-\frac{1}{2H}+, 4H+},}, & H \in (\frac{1}{2}, 1). \end{cases}$$

- When $b_1, b_2 \in L_t^{\frac{1}{1-H}} L_{loc}^{\frac{1}{1-H}}$ or $\mathbf{H}_{loc}^{1-\frac{1}{2H}+,2}$, we used the stopping time argument.

[3] J. Lehec, Representation formula for the entropy and functional inequalities. *Ann. Inst. Henri Poincaré Probab. Stat.*, 49 (2013).

Thank you!