Flow-distribution dependent SDEs and Navier-Stokes equations with fractional Brownian motion

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Probability representation of the NS equation

- Consider the following Navier-Stokes equation on $\mathbb{R}^d$ with $d = 2, 3$:

$$\begin{cases} \partial_t u = \Delta u + u \cdot \nabla u + \nabla p, \\ \mathrm{div} u = 0, \quad u_0 = \varphi, \end{cases} \tag{1}$$

- (Constantin-Iyer 2008, CPAM)

$$\begin{cases} X_t^x = x + \int_0^t u(s, X_s^x) \mathrm{d}s + \sqrt{2} W_t, \quad t \geq 0, \\ u(t, x) = \mathbf{P}\mathbb{E}[\nabla_x^{\mathrm{t}} Y_t^x \cdot \varphi(Y_t^x)], \end{cases} \tag{2}$$

where Y_t^x is the inverse of the flow mapping $x \to X_t^x$, ∇^{t} denotes the transpose of the Jacobi matrix $(\nabla X)_{ij} := \partial_{x_j} X^i$, and $\mathbf{P} := \mathbb{I} - \nabla \Delta^{-1} \mathrm{div}$ is the Leray projection.

Probability representation of the NS equation

- Question:

$$u(t,x) = \mathbf{P}\mathbb{E}[\nabla_x^{\mathrm{t}} Y_t^x \cdot \varphi(Y_t^x)] \overset{??}{=} B_X(t,x)$$

- Velocity & Vorticity:

$$w = \mathrm{curl} u = \begin{cases} \partial_2 u_1 - \partial_1 u_2, & d = 2; \\ \nabla \times u, & d = 3; \end{cases}$$

and

$$u = K_d * w, \;\; d = 2, 3,$$

with

$$K_2(x) := \frac{(x_2, -x_1)}{2\pi |x|^2}, \; K_3(x)h = \frac{x \times h}{4\pi |x|^3}. \qquad (3)$$

Probability representation of the NS equation

- (Zhang 2016, AoAP)

$$w(t,x) = \begin{cases} \mathbb{E}\left((\mathrm{curl}\varphi)(Y_t^x)\det(\nabla Y_t^x)\right), & d=2, \\ \mathbb{E}\left(\nabla_x^{\mathrm{t}} Y_t^x \cdot (\mathrm{curl}\varphi)(Y_t^x)\right), & d=3. \end{cases}$$

and

$$u(t,x) = \begin{cases} \mathbb{E}\left(\int_{\mathbb{R}^2} K_2(x - X_t^y)\cdot(\mathrm{curl}\varphi)(y)\mathrm{d}y\right), & d=2, \\ \mathbb{E}\left(\int_{\mathbb{R}^3} K_3(x - X_t^y)\cdot \nabla X_t^y \cdot (\mathrm{curl}\varphi)(y)\mathrm{d}y\right), & d=3. \end{cases}$$

Flow-distributional dependent SDEs

- $d = 2$, we define

$$B(x, \mu^{\cdot}) := \int_{\mathbb{R}^2} (K_2 * \mu^y)(x) \mathrm{curl}\varphi(y) \mathrm{d}y.$$

  Then X_t^x solves the following closed SDE:

$$X_t^x = x + \int_0^t B(X_s^x, \mu_s^{\cdot}) \mathrm{d}s + \sqrt{2} W_t. \tag{4}$$

- When $\mathrm{curl}\varphi \in \mathscr{P}(\mathbb{R}^2)$, flow-distributional dependent SDE (FDSDE) (4) can induce the following distributional dependent SDE (DDSDE):

$$X_t = X_0 + \int_0^t (K_2 * \mu_s)(X_s) \mathrm{d}s + \sqrt{2} W_t, \quad X_0 \overset{(d)}{=} \mathrm{curl}\varphi(y) \mathrm{d}y, \tag{5}$$

  by letting

$$\mathbb{P} \circ (X_\cdot)^{-1} := \int_{\mathbb{R}^2} \mathbb{P} \circ (X_\cdot^y)^{-1} \mathrm{curl}\varphi(y) \mathrm{d}y.$$

Flow-distributional dependent SDEs

- The FDSDE (4) was introduced by [Chorin 1973, JFM] as the random vortex method to simulate viscous incompressible fluid flows for smooth kernels.
- [Beale-Majda 1981, MoC], [Marchioror-Pulvirenti 1982, CMP], [Goodman 1987, CPAM], [Long 1988, JAMS].
- Propagation of chaos for interaction particle system: [Jabin-Wang 2018, Invent], [Feng-Wang 2023], [Wang 2024]; [Wang-Zhao-Zhu 2024, ARMA]......
- Moderately interacting particle systems: [Flandoli-Olivera-Simon 2020, SIAM J. MATH. ANAL], [Olivera-Richard-Tomašević 2021]......
- Well-posedness of DDSDE (5): [Zhang 2023, CMS], [Chaudru de Raynal-Jabir-Menozzi, 2023], [Barbu-Röckner-Zhang, 2023], [H.-Röckner-Zhang 2024, AoP]......
- (Question:) Well-posedness of FDSDE (4)?

Flow-distributional dependent SDEs

- When $d = 3$, we introduce a matrix-valued process $U_t^x := \nabla X_t^x$. Then U solves the following linear ODE:

$$U_t^x = \mathbb{I}_{3\times 3} + \int_0^t \bar{\mathbb{E}}\left(\int_{\mathbb{R}^3} \nabla K_3(X_s^y - \bar{X}_s^y)\cdot \bar{U}_s^y \cdot (\mathrm{curl}\varphi)(y)\mathrm{d}y\right)\mathrm{d}s,$$

  where $\bar{U}$ is an independent copy.
- Let $(\mu^x)_{x\in\mathbb{R}^3}$ be a family of probability measures over $\mathbb{R}^3 \times \mathcal{M}^3$, where $\mathcal{M}^3$ stands for the space of all 3×3-matrices. Now let us introduce

$$B(x,\mu) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3\times\mathcal{M}^3} K_3(x-z)\cdot M\mu^y(\mathrm{d}z\times\mathrm{d}M)\cdot(\mathrm{curl}\varphi)(y)\mathrm{d}y.$$

- Then we obtain the following closed FDSDE

$$\begin{cases} X_t^x = x + \int_0^t B(X_r^x, \mu_r^\cdot)\mathrm{d}r + \sqrt{2}W_t, \\ U_t^x = \mathbb{I}_{3\times 3} + \int_0^t \nabla B(\cdot, \mu_r^\cdot)(X_r^x)U_r^x \mathrm{d}r, \end{cases}$$

  where $\mu_t^x := \mathbb{P}\circ(X_t^x, U_t^x)^{-1} \in \mathscr{P}(\mathbb{R}^3 \times \mathcal{M}^3)$ for $x \in \mathbb{R}^3$.

Probability representation of the NSE-backward form

- On the other hand, setting $\tilde{u}(t,x) := u(T-t,x)$ and $\tilde{p}(t,x) := p(T-t,x)$, then $\tilde{u}$ solves the following backward Navier-Stokes equation:

$$\begin{cases} \partial_t \tilde{u} + \Delta \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} + \nabla \tilde{p} = 0, \\ \mathrm{div} \tilde{u} = 0, \quad \tilde{u}_T = \varphi. \end{cases}$$

- (Zhang 2010, PTRF)

$$\begin{cases} \tilde{X}^x_{s,t} = x + \int_s^t \tilde{u}(r, \tilde{X}^x_{s,r}) \mathrm{d}r + \sqrt{2}(W_t - W_s), \quad (s,t) \in \mathbb{D}_T, \\ \tilde{u}(t,x) = \mathbf{P}\mathbb{E}[\nabla^{\mathrm{t}} \tilde{X}^x_{t,T} \cdot \varphi(\tilde{X}^x_{t,T})]. \end{cases} \tag{6}$$

Backward flow-distributional dependent SDEs

- Similarly, (6) can be transformed into the following backward FDSDE:

$$\tilde{X}^x_{s,t} = x + \int_s^t \tilde{B}(\tilde{X}^x_{s,r}, \tilde{\mu}^{\cdot}_{r,T})\mathrm{d}r + \sqrt{2}(W_t - W_s), \tag{7}$$

where $\mu^x_{s,t}$ is the law of $X^x_{s,t}$, and

$$\tilde{B}(x, \mu^{\cdot}) = K_2 * \left(\int_{\mathbb{R}^2} \mathrm{curl}\varphi(y)\mu^{\cdot}(\mathrm{d}y) \right)(x).$$

- Recall the previous drift B in forward FDSDE (4):

$$B(x, \mu^{\cdot}) := \int_{\mathbb{R}^2} (K_2 * \mu^y)(x)\mathrm{curl}\varphi(y)\mathrm{d}y.$$

Forward & backward FDSDEs

(i) Forward FDSDE:

$$X_t^x = x + \int_0^t B(r, X_r^x, \mu_r^{\boldsymbol{\cdot}})\mathrm{d}r + \int_0^t \Sigma(r, X_r^x, \mu_r^{\boldsymbol{\cdot}})\mathrm{d}W_r, \quad t \in [0, T]. \tag{8}$$

(ii) Backward FDSDE:

$$\tilde{X}_{s,t}^x = x + \int_s^t \tilde{B}(r, \tilde{X}_{s,r}^x, \tilde{\mu}_{r,T}^{\boldsymbol{\cdot}})\mathrm{d}r + \int_s^t \Sigma(r, \tilde{X}_{s,r}^x, \tilde{\mu}_{r,T}^{\boldsymbol{\cdot}})\mathrm{d}W_r, \quad (s,t) \in \mathbb{D}_T. \tag{9}$$

- ▷ (Stroock-Varadhan): Weak solution $\iff$ Martingale solution;
- ▷ (Barlow): Uniqueness in law $\not\Rightarrow$ Existence of strong solution.
- ▷ (Shaposhnikov-Wresch, Anzeletti): Many counterexamples.

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- ▷ Shaposhnikov, A. and Wresch, L. (2022). Pathwise vs. path-by-path uniqueness. *Ann. Inst. Henri Poincaré Probab. Stat.*
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SDEs and PDEs

- Consider the following SDE:

$$X_{s,t}(x) = x + \int_s^t b(r, X_{s,r}(x))\mathrm{d}r + \sqrt{2}(W_t - W_s);$$

- Forward Fokker-Planck equation (FPE):

$$\partial_t \mu_{s,t} = \Delta \mu_{s,t} - \mathrm{div}(b(t)\mu_{s,t}), \quad \mu_{s,s} = \delta_x;$$

- Backward Fokker-Planck-Kolmogorov equation (BKE):

$$\partial_s u_{s,t} + \Delta u_{s,t} + b(s) \cdot \nabla u_{s,t} + f = 0, \quad u_{t,t} = \varphi.$$

SDEs and PDEs

$$u_{s,t}(x) = \mathbb{E}\varphi(X_{s,t}(x)) + \mathbb{E}\int_s^t f(r, X_{s,r})\mathrm{d}r$$

↑ Itô's formula to $r \to u_{r,t}(X_{s,r}(x))$

$$X_{s,t}(x)$$

↓ Itô's formula to $r \to \phi(X_{s,r}(x))$ for any $\phi \in C_b^2$

$$\mathbb{P} \circ (X_{s,t}(x))^{-1} \text{ satisfies (FPE)}$$

What if b is not a function?

- Brox diffusion (white noise); $b = \nabla U$ with some Hölder potential; Other noises.
- (Weak solution):
  - $$A_t^b := \lim_{n\to\infty} \int_0^t b_n(s, X_s)\mathrm{d}s \quad \text{exists}$$

    and $X_t = X_0 + A_t^b + W_t$.

What if b is not a function?

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- (Weak solution):
  - $$A_t^b := \lim_{n\to\infty} \int_0^t b_n(s, X_s)\mathrm{d}s \quad \text{exists}$$

    and $X_t = X_0 + A_t^b + W_t$.
- (Martingale solution):
  - For any $f \in \mathbf{C}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, consider the related PDE

    $$\partial_t u + \Delta u + b \cdot \nabla u + f = 0, \quad u(T) = 0.$$

    We call $\mathbb{P} \in \mathscr{P}(\mathcal{C}_T)$ a martingale solution if

    $$u(t, \omega_t) - u(t, \omega_0) - \int_0^t f(r, \omega_r)\mathrm{d}r \quad \text{is a } \mathbb{P}\text{-martingale}.$$
  - N. Ethier and G. Kurtz. Markov Processes: Characterization and Convergence. *Wiley series in probability and mathematical statistic. Wiley*, 1986.

Scale analysis

- Let $\dot{\mathbf{H}}_p^\alpha$ be the homogenous Bessel potential space, where $\alpha \leq 0$ and $p \in [1, \infty]$ and suppose for some $q \in [1, \infty]$

$$b \in L^q(\mathbb{R}_+; \dot{\mathbf{H}}_p^\alpha),$$

and SDE (**??**) admits a solution denoted by X. For $\lambda > 0$, we define

$$X_t^\lambda := \lambda^{-1} X_{\lambda^2 t}, \quad W_t^\lambda := \lambda^{-1} W_{\lambda^2 t}, \quad b^\lambda(t, x) := \lambda b(\lambda^2 t, \lambda x).$$

- Then we have

$$\mathrm{d}X_t^\lambda = b^\lambda(t, X_t^\lambda)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t^\lambda,$$

where

$$\|b^\lambda\|_{L^q(\mathbb{R}_+; \dot{\mathbf{H}}_p^\alpha)} = \lambda^{1+\alpha-\frac{d}{p}-\frac{2}{q}} \|b\|_{L^q(\mathbb{R}_+; \dot{\mathbf{H}}_p^\alpha)}.$$

- As $\lambda \to 0$,

Subcritical: $\frac{d}{p} + \frac{2}{q} < 1 + \alpha$; **Critical**: $\frac{d}{p} + \frac{2}{q} = 1 + \alpha$;
Supercritical: $\frac{d}{p} + \frac{2}{q} > 1 + \alpha$.

A well-defined restriction on α

- Consider the related PDE:

$$\partial_t u = \Delta u + b \cdot \nabla u + f.$$

- Assume $b \in \mathbf{C}^\alpha$ with the differentiability index $\alpha < 0$.
- According to the Schauder theory of the heat equation, $u \in \mathbf{C}^{2+\alpha}$.
- To make the product $b \cdot \nabla u$ meaningful, we need to stipulate that $1 + 2\alpha > 0$, which implies $\alpha > -\frac{1}{2}$.
  - (Delarue-Diel 2016) rough path & (Cannizzaro-Chouk 2018) para-controlled calculus: $b \in \mathbf{C}^{-2/3+}$ is some Gaussian noise.
  - (Question) Arbitrary function b? $\alpha \to -1$?

Well-known results

SEU: Strong existence-uniqueness; WEU: Weak existence-uniqueness;
WE: Weak existence; EUP: Existence-uniqueness of path-by-path solution.

Value of α	Subcritical	Critical	Supercritical
$\alpha = 0$	SEU: $\mathrm{V}^{79}_{[1]}$, $\mathrm{KR}^{05}_{[2]}$, $\mathrm{Z}^{05,10}_{[3,4]}$ EUP: $\mathrm{D}^{07}_{[5]}$, $\mathrm{ALL}^{23}_{[6]}$	WEU&SEU: $\mathrm{BFGM}^{19}_{[7]}$, $\mathrm{K}^{21}_{[8]}$, $\mathrm{RZ}^{21}_{[9]}$, $\mathrm{KM}^{23}_{[10]}$	WE: $\mathrm{ZZ}^{21}_{[11]}$
$\alpha \in [-\frac{1}{2}, 0)$	WEU: $\mathrm{BC}^{01}_{[12]}$, $\mathrm{FIR}^{17}_{[13]}$, $\mathrm{ZZ}^{17}_{[14]}$	–	–
$\alpha \in [-1, -\frac{1}{2})$	–	–	–

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[5] A. M. Davie. *Int. Math. Res. Not. IMRN* **24**. [6] L. Anzeletti, K. Lê and C. Ling. arXiv:2304.06802.
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Zvonkin's transformation- a method to kill the drift

- Consider the following BKE:

$$\partial_t \Phi + \Delta \Phi + b \cdot \nabla \Phi = 0, \quad \Phi(T,x) = x,$$

where $\Phi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$. We assume that if we can use Itô's formula to $s \to \Phi(s, X_s)$ and then

$$\mathrm{d}\Phi(t, X_t) = \sqrt{2}\nabla \Phi(t, X_t)\mathrm{d}W_t.$$

- We assume that $\Phi(t, \cdot)$ is an C^1-diffeomorphism.
- We define $(Y_t)_{t\geq 0} := (\Phi(t, X_t))_{t\geq 0}$ and note that $(Y_t)_{t\geq 0}$ satisfies the SDE without drift.

Weak well-posedness of subcritical SDEs with $\alpha \in (-1, -\frac{1}{2})$

Main results

($\mathbf{H}^{\rm sub}$) Let $(\alpha,p,q)\in(-1,-\frac{1}{2}]\times[2,\infty)^2$ with $\frac{d}{p}+\frac{2}{q}<1+\alpha$. Suppose that

$$\kappa_1^b:=\|b\|_{\mathbb{L}^q_T\mathbf{B}^{\alpha}_{p,q}}<\infty\quad\text{and}\quad\kappa_2^b:=\|\mathrm{div} b\|_{\mathbb{L}^q_T\mathbf{B}^{-2-\alpha}_{p,q/(q-1)}}<\infty.$$

Theorem 1 (H.-Zhang 2023)

*Under the condition (**$H^{\rm sub}$**), there is unique weak solution to SDE* (**??**). *Moreover, $t\to A^b_t$ has finite p-variation with some $p<2$.*

Main results

($\mathbf{H}^{\rm sub}$) Let $(\alpha, p, q) \in (-1, -\frac{1}{2}] \times [2, \infty)^2$ with $\frac{d}{p} + \frac{2}{q} < 1 + \alpha$. Suppose that

$$\kappa_1^b := \|b\|_{\mathbb{L}_T^q \mathbf{B}_{p,q}^{\alpha}} < \infty \quad \text{and} \quad \kappa_2^b := \|{\rm div} b\|_{\mathbb{L}_T^q \mathbf{B}_{p,q/(q-1)}^{-2-\alpha}} < \infty.$$

Theorem 1 (H.-Zhang 2023)

Under the condition ($\mathbf{H}^{\rm sub}$), there is unique weak solution to SDE (**??**). *Moreover, $t \to A_t^b$ has finite p-variation with some $p < 2$.*

- Suppose that $b \in \mathbb{L}_T^q \mathbf{B}_{p,1}^{-1/2}$ with $\frac{d}{p} + \frac{2}{q} < \frac{1}{2}$. Then **($\mathbf{H}^{\rm sub}$)** holds for $\alpha = -\frac{1}{2}$. Moreover, when ${\rm div} b = 0$, **($\mathbf{H}^{\rm sub}$)** holds.
- For any Lipschitz function $g : \mathbb{R}^d \to \mathbb{R}$,

$$\int_0^t g(X_s) {\rm d}A_s^b \quad \text{is a Young integral.}$$

Example:Gaussian noises

- For given $\gamma \in (d-2, d)$, we define the Gaussian noise b by the following covariance

$$\mathbb{E}b(f)b(g) = \int_{\mathbb{R}^d} \hat{f}(\xi)\hat{g}(-\xi)|\xi|^{-\gamma}\Big(\mathbb{I}_{d\times d} - \frac{\xi\otimes\xi}{|\xi|^2}\Big)\mathrm{d}\xi.$$

- Then we have for almost surely ω

$$b(\omega,\cdot) \in \cap_{p\in[1,\infty)}\mathbf{B}^{-1+}_{p,loc}(\mathbb{R}^d) \quad \mathrm{div} b(\omega) = 0.$$

Sketch of the proof

- Consider the following BKE:

$$\partial_t u + \Delta u + b \cdot \nabla u + f = 0, \quad u(T) = 0, \quad t \in [0, T].$$
$$b \in \mathbf{C}^{\alpha}, \quad u \in \mathbf{C}^{2+\alpha}.$$

- We define $b \cdot \nabla u := b \odot \nabla u + \mathrm{div} b \circ u + \mathrm{div} b \prec u$ where

$$\boxed{b \odot \nabla u := \mathrm{div}(b \prec u + b \circ u) + b \succ \nabla u.}$$

- The paraproduct implies that

$$\|\mathrm{div} b \circ u + \mathrm{div} b \prec u\|_{\alpha} \lesssim \|\mathrm{div} b\|_{-2-\alpha} \|u\|_{2+\alpha}$$

and

$$\begin{aligned} \|b \odot \nabla u\|_{\alpha} &\lesssim \|b \prec u + b \circ u\|_{1+\alpha} + \|b\|_{\alpha} \|\nabla u\|_{\mathbb{L}^{\infty}} \\ &\lesssim \|b\|_{\alpha} (\|u\|_{1} + \|\nabla u\|_{\mathbb{L}^{\infty}_{T}}) \lesssim \|b\|_{\alpha} \|u\|_{2+\alpha}. \end{aligned}$$

Sketch of the proof

- Consider the following BKE:

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- Therefore, we have $u \in \mathbf{C}^{2+\alpha}$ and

$$\lim_{\delta \to 0} \sup_{|t-s| \leq \delta, t,s \in [0,T]} \|\nabla u(t) - \nabla u(s)\|_{L^{\infty}} = 0.$$

- Zvonkin's transformation: taking $f = b$ and $\Phi_t(x) := x + u(t,x)$.

Weak solutions to supercritical SDEs with $\alpha = -1$

The setting

- **($\mathbf{H}^{sup}$)** We assume $d \geq 2$, $b \in L_T^q \mathbf{H}_p^{-1}$ with $p, q \in [2, \infty]$,
$$\frac{d}{p} + \frac{2}{q} < 1, \quad \mathrm{div} b = 0.$$

The setting

- $(\mathbf{H}^{sup})$ We assume $d \geq 2$, $b \in L^q_T \mathbf{H}^{-1}_p$ with $p, q \in [2, \infty]$,

$$\frac{d}{p} + \frac{2}{q} < 1, \quad \mathrm{div} b = 0.$$

- Let $b_n \in \mathbf{C}^\infty_b(\mathbb{R}_+ \times \mathbb{R}^d)$ with $\lim_{n\to\infty} \|b_n - b\|_{L^q_T \mathbf{H}^{-1}_p} = 0$ and consider the following approximating SDE

$$X^n_t = X_0 + \int_0^t b_n(s, X^n_s) \mathrm{d}s + \sqrt{2} W_t.$$

- We denote the distribution of $(X^n_t)_{t\in[0,T]}$ by $\mathbb{P}_n \in \mathscr{P}(C([0,T]; \mathbb{R}^d))$.

Main results

Theorem 2 (H.-Zhang 2023)

i) For any $\mathscr{F}_0$ *measurable random variable* X_0, $\{\mathbb{P}_n\}_{n=1}^{\infty}$ *is* **tight** *in* $\mathscr{P}(C([0,T];\mathbb{R}^d))$.

ii) Moreover, if the distribution of X_0 *has an* L^2 *density w.r.t. the Lebesgue measure, then there is a continuous process* $(X_t)_{t\in[0,T]}$ *such that*

$$X_t = X_0 + \lim_{n\to\infty}\int_0^t b_n(r,X_r)\mathrm{d}r + \sqrt{2}W_t,$$

where the limit here is taken in $L^2(\Omega)$.

iii) Let $\mathbb{P}$ *be the law of the solution* $(X_t)_{t\in[0,T]}$*. The following Markov property holds:*

$$\mathbb{E}_{\mathbb{P}}[f(\omega_t)|\mathscr{B}_s] = \mathbb{E}_{\mathbb{P}}[f(\omega_t)|\omega_s], \quad 0\le s\le t\le T,\ f\in \mathbf{C}_b(\mathbb{R}^d).$$

- When $b\in L^2([0,T]\times\mathbb{R}^d)$ or $b\in L_T^\infty\mathbf{B}^{-1}_{\infty,2}$ (critical & ill-defined), there is only one accumulation point of $\{\mathbb{P}_n\}_{n=1}^{\infty}$. That is for any $b_n\to b$, $\mathbb{P}_n$ converges to the distribution of $(X_t)_{t\in[0,T]}$.

Example: Particle system with singular kernels

- Consider the following singular interaction particle system in $\mathbb{R}^{Nd}$:

$$\mathrm{d}X_t^{N,i} = \sum_{j\neq i} \gamma_j K(X_t^{N,i} - X_t^{N,j})\mathrm{d}t + \sqrt{2}\mathrm{d}W_t^{N,i}, \quad i = 1, \cdots, N, \tag{10}$$

  where $K \in \mathbf{H}_\infty^{-1}(\mathbb{R}^d; \mathbb{R}^d)$ is divergence free, $W_t^{N,i}, i = 1, \cdots, N$ are N-independent standard d-dimensional Brownian motions, $\gamma_j \in \mathbb{R}$ and initial value has an L^2-density.
- (Jabin-Wang 2018) Existence of the related FPE and propagation of chaos. (The existence of a solution to the SDE (10) appears to be open).
- As a result, we have the weak existence to the N-particle system SDE (10).

Example: GFF and super-diffusive

- Let $d = 2$, $\varepsilon \in (0,1]$ and b_ε be a Gaussian field with

$$\mathbb{E}b_\varepsilon(f)b_\varepsilon(g) = \int_{|\xi|\leq 1/\varepsilon} \hat{f}(\xi)\hat{g}(-\xi)\Big(\mathbb{I}_{d\times d} - \frac{\xi\otimes\xi}{|\xi|^2}\Big)\mathrm{d}\xi.$$

- When $\varepsilon \to 0$, $b := \lim_\varepsilon b_\varepsilon$ formally satisfies

$$b := \nabla^\perp \xi := (-\partial_{x_2}\xi_1, \partial_{x_1}\xi_2) \in \mathbf{C}^{-1-} \quad \mathrm{div} b = 0,$$

  where $\xi = \xi(x)$ is the two-dimensional Gaussian Free Field (GFF)
- (Super-diffusive)
  When $\varepsilon = 1$, $\mathbb{E}|X_t|^2 \asymp t\sqrt{\ln t}$
  (Cannizzaro-HaunschmidSibitz-Toninelli 2022)
  (Chatzigeorgiou-Morfe-Otto-Wang 2022).
- For any $p \in (2,\infty)$

$$\sup_{\varepsilon<1/2} \Big\| \frac{b_\varepsilon}{\sqrt{\ln \varepsilon}} \Big\|_{\mathbf{H}^{-1}_{p,loc}} < \infty, \quad a.s.$$

  By our results, one sees that the solutions $\{X^\varepsilon_t\}_{[0,T]}$ to the following approximation SDEs is tight

$$\mathrm{d}X^\varepsilon_t = \frac{b_\varepsilon(X^\varepsilon_t)}{\sqrt{\ln \varepsilon}}\mathrm{d}t + \sqrt{2}\mathrm{d}W_t.$$

Sketch of the proof– Tightness

- Consider the following backward PDE

$$\partial_t u + \Delta u + b \cdot \nabla u + f = 0, \quad u(T) = 0 \qquad \text{(PDE)}$$

and the following approximation PDEs

$$\partial_t u_n + \Delta u_n + b_n \cdot \nabla u_n + f = 0, \quad u(T) = 0 \qquad \text{(APDE)}$$

- Under the condition $(\mathbf{H}^{sup})$, by De Giorgi's method in (Zhang-Zhao 2021), we have

$$\sup_n \left(\|u_n\|_\infty + \|\nabla u_n\|_2 \right) < \infty,$$

which implies the there is a weak solution u to (PDE).

- (Problem): Since we don't know whether $\langle u, b \cdot \nabla u \rangle = 0$ holds a priority, we don't have the uniqueness of (PDE).

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- By Itô's formula,

$$\sup_n \left| \mathbb{E} \int_0^T f(r, X_r^n) \mathrm{d}r \right| \leq \|u_n\|_\infty \lesssim \|f\|_{L_T^q \mathbf{H}_p^{-1}} \quad \text{(1st Krylov estimate).}$$

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- By Aldous' criterion of tightness and the strong Markov property, we only need to show

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$$\begin{aligned}\mathbf{E}|X^n_\tau-x_0|\leqslant \mathbf{E}h_\varepsilon(X^n_\tau)&=\varepsilon+\mathbf{E}\left(\int_0^\tau(\Delta+b_n(s)\cdot\nabla)h_\varepsilon(X^n_s)\mathrm{d}s\right)\\&\lesssim \varepsilon+\delta\varepsilon^{-1}+\left|\mathbf{E}\left(\int_0^\tau(b_n\cdot\nabla h_\varepsilon)(s,X^n_s)\mathrm{d}s\right)\right|\\&\overset{1stKE}{\lesssim}\ \varepsilon+\delta\varepsilon^{-1}+\|b_n\cdot\nabla h_\varepsilon\|_{\mathbb{L}^q_\delta\mathbf{H}^{-1}_p}\left(\lesssim\|b_n\|_{\mathbb{L}^q_\delta\mathbf{H}^{-1}_p}\|\nabla h_\varepsilon\|_{\mathbf{C}^1_b}\right)\\&\lesssim \varepsilon+\delta\varepsilon^{-1}+\|b\|_{\mathbb{L}^q_\delta\mathbf{H}^{-1}_p}\quad\to 0\end{aligned}$$

as $\delta\to 0$ and $\varepsilon\to 0$.

Sketch of the proof– Weak existence

- Tightness + Skorokhod's representation theorem $\Rightarrow$ limit process $(X_t)_{t\in[0,T]}$.
- What we need : $\lim_{n\to\infty}\sup_{m\geq n}\mathbb{E}|\int_0^t(b_n-b_m)(s,X_s)\mathrm{d}s|=0$.

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$$\partial_s u_n + \Delta u_n + b_n \cdot \nabla u_n + f = 0, \quad u(t) = 0$$

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  - ▷ By the representation of the solution to BKE,

$$\begin{aligned}I_n(f)&=2\mathbb{E}\int_0^t\int_s^t f(s,X_s^n)f(r,X_r^n)\mathrm{d}r\mathrm{d}s\\&=2\mathbb{E}\int_0^t f(s,X_s^n)\mathbb{E}^{\mathscr{F}_s}\left[\int_s^t f(r,X_r^n)\mathrm{d}r\right]\mathrm{d}s\\&=2\mathbb{E}\int_0^t f(s,X_s^n)u_n(s,X_s^n)\mathrm{d}s=2\int_0^t\langle f(s)u_n(s),\rho_n(s)\rangle\mathrm{d}s\\&\lesssim\|f\|_{L_T^q\mathbf{H}_p^{-1}}\|u_n\|_{L_t^\infty L^2\cap L_T^2\mathbf{H}_2^1}\|\rho_n\|_{L_T^\infty L^2\cap L_T^2\mathbf{H}_2^1}\lesssim\|f\|^2_{L_T^q\mathbf{H}_p^{-1}}\|\rho_0\|_2.\end{aligned}$$

Sketch of the proof–Markov property

- Idea: obtain the uniqueness martingale solution.

Definition 3 (Martingale solution)

Let $\mu \in \mathscr{P}(\mathbb{R}^d)$. We call a probability measure $\mathbb{P} \in \mathscr{P}(\mathbb{C}_T)$ a martingale solution of SDE (**??**) starting from μ, if $\mathbb{P} \circ (\omega_0)^{-1} = \mu$ and for any $f \in C_c^\infty([0,T] \times \mathbb{R}^d)$,

$$M_t^f := u(t, \omega_t) - u(0, \omega_0) - \int_0^t f(r, \omega_r) \mathrm{d}r, \;\; \omega_\cdot \in \mathbb{C}_T,$$

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- We couldn't show the existence of a solution to the martingale solution such that the definition holds for all solutions u.
- We can find a bounded linear operator

$$\boxed{\mathcal{S} : L_T^q \mathbf{H}_p^{-1} \to L_T^\infty L^\infty \cap L_T^2 \mathbf{H}_2^1}$$

  such that for any f, $u = \mathcal{S} f$ solves (PDE).
  - Once $b \in L_T^\infty \mathbf{B}_{\infty,2}^{-1}$, we have the uniqueness and stability for (PDE), which implies the uniqueness of the operator $\mathcal{S}$.

Sketch of the proof–Markov property

Definition 4 (Generalized martingale solution)

Let $\mu \in \mathscr{P}(\mathbb{R}^d)$. We call a probability measure $\mathbb{P} \in \mathscr{P}(\mathbb{C}_T)$ a generalized martingale solution of SDE (**??**) starting from μ and associated with the operator $\mathcal{S}$, if $\mathbb{P} \circ (\omega_0)^{-1} = \mu$ and for any $f \in C_c^\infty([0,T] \times \mathbb{R}^d)$,

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Theorem 3 (H.-Zhang 2023)

Assume μ has an L^2 density w.r.t. the Lebesgue measure. There is a unique generalized martingale solution w.r.t. the $\mathcal{S}$.

- We can find a subsequence $\{n_k\}_{k=1}^\infty$ such that $u_{n_k} \to \mathcal{S}f$ ($\mathcal{S}$ depends on this subsequence). Then the law of a weak solution is just a generalized martingale solution. The Markov property follows from the definition of the generalized martingale solution.

Further works

- Uniqueness in the supercritical cases.
- Characterize the limit of the approximation solutions to the SDEs with drift $b = \nabla^{\perp}\text{GFF}$.
- RDEs with "singular" diffusion coefficients.
- $\cdots$

Thank you!

Danke!