The 18th Workshop on Markov Process and Related Topics

SDEs with supercritical distributional drifts and RDEs with subcritical drifts

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Based on a joint work with Xicheng Zhang, Rongchan Zhu and Xiangchan Zhu

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Regularization by noise

- Consider the following SDE:

$$X_t = X_0 + \int_0^t b(s, X_s)\mathrm{d}s + \sqrt{2}W_t,$$

where $(W_t)_{t\geq 0}$ is a standard d-dimensional Brownian motion.

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where $(W_t)_{t\geq 0}$ is a standard d-dimensional Brownian motion.
- (ODE): $b \in \mathbf{C}^\alpha(\alpha < 1) \Rightarrow$ non-uniqueness;
- (SDE): Well-posedness
  - (Zvonkin 1974): b is bounded and Dini continuous;
  - (Veretennikov 1979): b is bounded;
  - (Krylov-Röckner 2005): $b \in L^q_T L^p_x$ with $d/p + 2/q < 1$;
  - (Röckner-Zhao 2022, Krylov 2021-2023): $b \in L^q_T L^p_x$ with $d/p + 2/q = 1$;
  - (Zhang, Xie, Zhao, Xia,...): Multiplicative noise cases.
- Methods: Relation between the SDEs and the Kolmogorov PDEs
  (Zvonkin's transformation, Itô-Tanaka's trick,...)

Motivations

- **Navier-Stokes equations**

$$\partial_t u = \Delta u + u \cdot \nabla u + \nabla p = 0, \quad \mathrm{div} u = 0.$$

  - (Constantin-Iyer 2008, Zhang 2012,...)

$$\begin{cases} X_t^x = x + \int_0^t u(s, X_s^x)\mathrm{d}s + \sqrt{2}W_t, \quad t \geq 0, \\ u(t,x) = \mathbf{P}\mathbb{E}[\nabla^T (X_t^\cdot)^{-1}(x)\phi((X_t^\cdot)^{-1}(x))]. \end{cases}$$

  - $u \in L_t^q L_x^p$ with $\frac{d}{p} + \frac{2}{q} = \frac{d}{2}$.
  - Existence of the solution to SDE: (Zhang-Zhao 2021) $d/p + 2/q < 2$ and $\mathrm{div} u = 0$.

- ***N*-particle systems**

$$\mathrm{d}X_t^{N,i} = \frac{1}{N}\sum_{j=1}^N K(X_t^{N,i} - X_t^{N,j})\mathrm{d}t + \mathrm{d}W_t^i,$$

  - $K(x)$: Biot-Savart law, Coulomb potential, ...
  - $K(x) \asymp |x|^{-d+1}$.

Motivations-distributional type

- Consider the following SDE:

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \sqrt{2}W_t.$$

- **Gradient flow**
  - (Bass-Chen 2001) $b = \nabla B$ with some $B \in \mathbf{C}^\beta$ and $\beta \in (0,1)$.
  - (Flandoli-Issoglio-Russo 2017, Zhang-Zhao 2018) $b \in H^{-\alpha,p}$ with $\alpha \in [0, \frac{1}{2}]$ and $\alpha + \frac{d}{p} < 1$.
- **Brox diffusion** (Sinai 1982, Brox 1986)
  - $b = \xi \in \mathbf{C}^{-1/2-}$ is one-dimensional spatial white noise.
  - (Delarue-Diel 2016) rough path & (Cannizzaro-Chouk 2018) paracontrolled calculus: $b \in \mathbf{C}^{-2/3+}$ is some Gaussian noise.......
- **Super-diffusive**
  - $b = \mathbf{C}^{-1-}$ (well-posedness is open, even does not hold) and $\mathbb{E}|X_t|^2 \asymp t\sqrt{\ln t}$.
  - (Chatzigeorgiou-Morfe-Otto-Wang 2022), (Feltes-Weber 2022), ...

Scaling and conditions

- For any $\varepsilon > 0$, we define $\tilde{W}_t := \varepsilon^{-1} W_{\varepsilon^2 t}$, $X_t^\varepsilon := \varepsilon^{-1} X_{\varepsilon^2 t}$ and have

$$X_t^\varepsilon = X_0^\varepsilon + \int_0^t b^\varepsilon(s, X_s)\mathrm{d}s + \sqrt{2}\tilde{W}_t,$$

  where $b^\varepsilon(t,x) = \varepsilon b(\varepsilon^2 t, \varepsilon x)$.
  - We note that for any $\alpha \geq 0$

$$\|b^\varepsilon\|_{L_t^q \dot{H}^{-\alpha,p}} = \varepsilon^{1-\alpha-\frac{d}{p}-\frac{2}{q}} \|b\|_{L_t^q \dot{H}^{-\alpha,p}}.$$

  - Conditions:

    - $\alpha + d/p + 2/q < 1$: subcritical;
    - $\alpha + d/p + 2/q = 1$: critical;
    - $\alpha + d/p + 2/q > 1$: supercritical.

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- We assume $b \in \mathbf{C}^{-\alpha}$ with some $\alpha \geq 0$ and consider the related PDE:

$$\partial_t u = \Delta u + b \cdot \nabla u + f \overset{(Schauder)}{\Rightarrow} u \in L_T^\infty \mathbf{C}^{2-\alpha}, \quad b \cdot \nabla u : \mathbf{C}^{-\alpha} \times \mathbf{C}^{1-\alpha}.$$

  - Conditions:
    - $\alpha < 1/2$: well-defined;
    - $\alpha \geq 1/2$: ill-defined.

Examples

- Consider the following SDE:

$$X_t = X_0 + \int_0^t b(s, X_s)\mathrm{d}s + \sqrt{2}W_t.$$

- Results and models:
  - (Krylov-Röckner 2005): $b \in L_T^q L_x^p$ with $d/p + 2/q < 1$ (sub-well);
  - (Röckner-Zhao 2022, Krylov 2021-2023): $b \in L_T^q L_x^p$ with $d/p + 2/q = 1$ (critical-well);
  - (Zhang-Zhao 2021): $b \in L_T^q L_x^p$ with $d/p + 2/q < 2$ (super-well);
  - (Flandoli-Issoglio-Russo 2017, Zhang-Zhao 2018):
    $b \in H^{-\alpha,p}$ with $\alpha \in [0, \frac{1}{2}]$ and $\alpha + \frac{d}{p} < 1$ (sub-well);
  - (Delarue-Diel 2016, Cannizzaro-Chouk 2018): $b \in \mathbf{C}^{-2/3+}$ is some Gaussian noise (sub-ill);
  - (Super-diffusive): $b \in \mathbf{C}^{-1-}$ (super-ill).

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  - (Super-diffusive): $b \in \mathbf{C}^{-1-}$ (super-ill).
- (Question) Whether we can obtain the solution to SDEs with supercritical & ill-defined conditions?

The setting

- We assume $d \geq 2$, $b \in L_T^q H^{-1,p}$ with

$$\frac{d}{p} + \frac{2}{q} < 1, \quad \mathrm{div} b = 0 \quad \text{(supercritical and ill-defined)}$$

and let $b_n \in \mathbf{C}_b^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ with $\lim_{n\to\infty} \|b_n - b\|_{L_T^q H^{-1,p}} = 0$. Consider the following approximating SDE

$$X_t^n = X_0 + \int_0^t b_n(s, X_s^n) \mathrm{d}s + \sqrt{2} W_t.$$

- We denote the distribution of $(X_t^n)_{t\in[0,T]}$ by $\mathbb{P}_n \in \mathcal{P}(C([0,T]; \mathbb{R}^d))$.

Main results

① **Theorem 1.1** (H.-Zhang-Zhu-Zhu 2023+)

For any $\mathscr{F}_0$ measurable random variable X_0, $\{\mathbb{P}_n\}_{n=1}^\infty$ is **tight** in $\mathcal{P}(C([0,T];\mathbb{R}^d))$. Moreover, if the distribution of X_0 has an L^r density w.r.t. the Lebesgue measure, where $1/r+1/p=1/2$, then there is a continuous process $(X_t)_{t\in[0,T]}$ such that

$$X_t = X_0 + \lim_{n\to\infty}\int_0^t b_n(r,X_r)\mathrm{d}r + \sqrt{2}W_t,$$

where the limit here is taken in $L^2(\Omega)$.

- When $b\in L^2([0,T]\times\mathbb{R}^d)$ or $b\in L_T^\infty\mathbf{B}_{\infty,2}^{-1}$ (critical & ill-defined), there is only one accumulation point of $\{\mathbb{P}_n\}_{n=1}^\infty$. That is for any $b_n\to b$, $\mathbb{P}_n$ converges to the distribution of $(X_t)_{t\in[0,T]}$.

Application

- Let $\xi = \xi(x)$ be a two-dimensional Gaussian Free Field (GFF) and ξ^{loc} be the cut-off of it with compact support.
- Define

$$b := \nabla^{\perp}\xi^{loc} := (-\partial_{x_2}\xi_1^{loc}, \partial_{x_1}\xi_2^{loc}) \in \mathbf{C}^{-1-} \quad \mathrm{div} b = 0.$$

- Let $b_\varepsilon := b * \phi_\varepsilon$. We have for any $p \in (2, \infty)$

$$\sup_{\varepsilon<1/2} \| \frac{b_\varepsilon}{\sqrt{\ln \varepsilon}} \|_{H^{-1,p}} < \infty, \quad a.s.$$

By our results, one sees that the solutions $\{X_t^\varepsilon\}_{[0,T]}$ to the following approximation SDEs is tight

$$\mathrm{d}X_t^\varepsilon = b_\varepsilon(X_t^\varepsilon)\mathrm{d}s + \sqrt{2}\mathrm{d}W_t.$$

Sketch of the proof

- Consider the following backward PDE

$$\partial_t u + \Delta u + b \cdot \nabla u + f = 0, \quad u(T) = 0. \tag{PDE}$$

- Formally, by Itô's formula, $\mathbb{E} \int_0^T f(r, X_r) \mathrm{d}r = \mathbb{E} u(0, X_0)$.

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  - By De Giorgi's method in (Zhang-Zhao 2021), we have

$$\left| \mathbb{E} \int_0^T f(r, X_r) \mathrm{d}r \right| \le \|u\|_{\mathbb{L}^\infty_T} \lesssim \|f\|_{L^q_T H^{-1,p}} \Rightarrow \text{tightness}.$$

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- (Problem): Since we don't know whether $\langle u, b \cdot \nabla u\rangle = 0$ holds a priority, we don't have the uniqueness of (PDE).
  - Once $b \in L^2([0,T] \times \mathbb{R}^d)$ or $b \in L^\infty_T \mathbf{B}^{-1}_{\infty,2}$, we have $\langle u, b \cdot \nabla u\rangle = 0$. Thus we have the uniqueness and stability.

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$$\left|\mathbb{E}\int_0^T f(r, X_r)\mathrm{d}r\right| \leq \|u\|_{\mathbb{L}_T^\infty} \lesssim \|f\|_{L_T^q H^{-1,p}} \Rightarrow \text{tightness.}$$

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- We can find a bounded linear operator

$$A : L_T^q H^{-1,p} \to L_T^\infty L^\infty \cap L_T^2 H^{1,2}$$

such that for any f, $u = Af$ solves (PDE).

Martingale solution

① **Theorem 1.2** (H.-Zhang-Zhu-Zhu 2023+)

There is a set $I \subset [0, T]$ containing 0 and T of full measure such that for any distribution μ_0 which has an L^2 density w.r.t. the Lebesgue measure, there is a unique measure $\mathbb{P} \in \mathcal{P}(C([0, T]; \mathbb{R}^d))$ such that for any $f \in C([0, T]; C_0(\mathbb{R}^d))$

$$M_t := Af(t, w_t) - Af(0, w_0) - \int_0^t f(r, w_r)\mathrm{d}r$$

is a $\mathbb{P}$-martingale on I and $\mathbb{P} \circ w_0^{-1} = \mu_0$.

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- When $b \in L^2([0,T] \times \mathbb{R}^d)$ or $b \in L_T^\infty \mathbf{B}_{\infty,2}^{-1}$, Af is the unique solution to (PDE).
- Uniqueness of PDE $\Rightarrow$ the unique accumulation point of $\{\mathbb{P}_n\}_{n=1}^\infty$.

Subcritical cases

② **Theorem 1.3** (H.-Zhang-Zhu-Zhu 2023+)

Assume that $b \in L^q_T H^{-\alpha,p}$ with some $\alpha \in (0,1)$, $\Gamma := \alpha + d/p + 2/q < 1$ and $\mathrm{div} b = 0$. For any $\mathscr{F}_0$ measurable random variable X_0, there is a unique (in law) continuous process $(X_t)_{t\in[0,T]}$ such that

$$X_t = X_0 + \lim_{n\to\infty} \int_0^t b_n(r, X_r)\mathrm{d}r + \sqrt{2}W_t,$$

where the limit here is taken in $L^2(\Omega)$, and for any $m \in \mathbb{N}$ and smooth functions f

$$\mathbb{E}\left|\int_s^t f(r, X_r)\mathrm{d}r\right|^m \leq C_m|t-s|^{(2-\Gamma)m/2}\|f\|^m_{L^q_T H^{-\alpha,p}}. \tag{1}$$

- Property (1) implies that $t \to \lim_{n\to\infty}\int_0^t b_n(r, X_r)\mathrm{d}r$ is a zero energy process.

Sketch of the proof

- In (PDE), we define

$$b \cdot \nabla u := \mathrm{div}(b \prec u + b \circ u) + b \succ \nabla u,$$

which by paraproduct implies that

$$\begin{aligned} \|b \cdot \nabla u\|_{L_T^q \mathbf{B}_{p,\infty}^{-\alpha}} &\lesssim \|b \prec u + b \circ u\|_{L_T^q \mathbf{B}_{p,\infty}^{1-\alpha}} + \kappa_b \|\nabla u\|_{\mathbb{L}_T^\infty} \\ &\lesssim \kappa_b (\|u\|_{L_T^\infty \mathbf{B}_{\infty,\infty}^1} + \|\nabla u\|_{\mathbb{L}_T^\infty}). \end{aligned}$$

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- Therefore, we have

$$\|u\|_{L^\infty_T \mathbf{B}^{2-\alpha-2/q}_{p,\infty}} \leq C \|f\|_{L^\infty_T \mathbf{B}^{-\alpha}_{p,\infty}} \quad \text{and} \quad \lim_{\delta \to 0} \sup_{|t-s| \leq \delta, t,s \in [0,T]} \|\nabla u(t) - \nabla u(s)\|_{L^\infty} = 0.$$

- Then we can construct the Zvonkin's transformation by taking $f = b$ and $\Phi_t(x) := x + u(t,x)$.

Rough path and RDE

- For a.s. $\omega \in \Omega$, the Brownian motion $W_t(\omega)$, which is in some probabaility space $(\Omega, \mathscr{F}, \mathbb{P})$, can be regarded as a rough path with

$$|W_t(\omega) - W_s(\omega)| \leq C|t-s|^{\alpha}, \quad |\mathbb{W}_{s,t}| := |\int_s^t (W_r - W_s) \mathrm{d}W_s(\omega)| \leq C|t-s|^{2\alpha}$$

for any $\alpha \in (1, 1/2)$. We denote $\mathbf{W} := (W, \mathbb{W})$.

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  for any $\alpha \in (1, 1/2)$. We denote $\mathbf{W} := (W, \mathbb{W})$.
- Then for any $\sigma = \sigma(x) \in \mathbf{C}_b^2$ the following rough differential equations (RDEs)

$$X_t = X_0 + \int_0^t \sigma(X_r)\mathrm{d}\mathbf{W}_r \tag{RDE}$$

  is well-defined by

$$\int_0^t \sigma(X_r)\mathrm{d}\mathbf{W}_r := \lim_{|\pi|\to 0} \sum_{r,s\in\pi} \left(\sigma(X_r)(W_s - W_r) + \sigma(X_r)\nabla\sigma(X_r)\mathbb{W}_{r,s}\right),$$

  where π is any partition of $[0, t]$.
- When $\sigma \in \mathbf{C}^3$, there is a unique solution to (RDE).

Main results

① **Theorem 2.1** (H.-Lê 2023+)

Let $T > 0$. Assume that $b \in L_T^q L^p$ with some $d/p + 2/q < 1$ and $p > 2$. There is a event $\Omega_{b,T}$ of full measure such that for any $\omega \in \Omega_{b,T}$ and $x \in \mathbb{R}^d$, there is a unique solution to the following RDE

$$X_t = x + \int_0^t b(r, X_r)\mathrm{d}r + \int_0^t \sigma(X_r)\mathrm{d}\mathbf{W}_r(\omega)$$

with

$$\int_0^t |b(r, X_r)|\mathrm{d}r < \infty.$$

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with

$$\int_0^t |b(r, X_r)|\mathrm{d}r < \infty.$$

- The uniqueness here is in path-by-path sense. That is for any fixed $\omega \in \Omega_{b,T}$ and solutions X_t and Y_t with $X_0 = Y_0$, we have $(X_t)_{t\in[0,T]} = (Y_t)_{t\in[0,T]}$ (no a.s.).
- The existence allows us to construct solution $(X_t)_{t\in[0,T]}$ with any $X_0(\omega)$, even if X_0 is not measurable.

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- The existence allows us to construct solution $(X_t)_{t\in[0,T]}$ with any $X_0(\omega)$, even if X_0 is not measurable.
- When $\sigma = \mathbb{I}$, this result has been obtained in (Anzeletti-Lê-Ling 2023).

Crucial of the proof

- Consider the following semi-flow

$$\phi_t^{s,x} = x + \int_s^t \sigma(\phi_r^{s,x}) \mathrm{d}\mathbf{W}_r.$$

- (Davie's estimate)

$$\mathbb{E}\left|\int_s^t (b(\phi_r^{s,x}) - b(\phi_r^{s,y}))dr\right|^m \lesssim |x-y|^m |t-s|^\alpha \|b\|^m_{L^p_q(T)}.$$

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- When $\sigma = \mathbb{I}$, in (Anzeletti-Lê-Ling 2023), $\phi_t^{s,x} = x + W_t - W_s$ and it is sufficient to show

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\left|\int_s^t \nabla b(x+W_r)dr\right|^m \lesssim |t-s|^\alpha \|b\|_{L_q^p(T)}^m.$$

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- (Method)
  The relation between the PDE and the SDE.

Further works

- Uniqueness in the supercritical cases.
- Characterize the limit of the approximation solutions to the SDEs with drift $b = \nabla^{\perp}\mathrm{GFF}$.
- RDEs with "singular" diffusion coefficients.
- $\cdots$

Thank you !