Second order fractional mean-field SDEs with singular kernels

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Background and motivation

N-particle systems

- Consider the following N-particle systems:

$$\begin{cases} \mathrm{d}X_t^{N,i} = V_t^{N,i}\mathrm{d}t, \\ \mathrm{d}V_t^{N,i} = \frac{1}{N}\sum_{j\neq i} K(X_t^{N,i} - X_t^{N,j})\mathrm{d}t + L_t^{(\alpha),i}. \end{cases}$$

  - $\frac{1}{N}$: mean-field scaling;
  - $K = \nabla U : \mathbb{R}^d \to \mathbb{R}^d$: interaction kernel with some potential U (e.g. $U(x) = |x|^{2-d}$, $\ln|x|$);
  - $\{L_t^{(\alpha),i}\}_{i=1}^{\infty}$ is a family of i.i.d. α-stable processes: collision and background medium.

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  - $\{L_t^{(\alpha),i}\}_{i=1}^{\infty}$ is a family of i.i.d. α-stable processes: collision and background medium.

- Plasma physics (Vlasov 1968, Carrillo-Choi-Salem 2019,...);
  Biosciences (Simon-Olivera 2018, Flandoli-Leimbach-Olivera 2019,...);
  . . .

Second order mean-field SDEs

- Propagation of chaos (Kac 1956, McKean 1967,..., Sznitman 1991, ..., Jabin-Wang 2016, 2018, Lacker 2018, 2021,...)
- $(X_t^{N,i}, V_t^{N,i})$ converges to the solution of the following mean-field SDEs:

$$\begin{cases} \mathrm{d}X_t = V_t \mathrm{d}t, \\ \mathrm{d}V_t = (K * \mu_{X_t})(X_t)\mathrm{d}t + \mathrm{d}L_t^{(\alpha)}, \end{cases}$$

where μ_{X_t} is the time marginal law of X_t.

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  where μ_{X_t} is the time marginal law of X_t.
- It can be rephrased as the following second order mean-field SDE:

$$\ddot{X}_t = (K * \mu_t)(X_t) + \dot{L}_t^{(\alpha)}.$$

Nonlinear Fokker-Planck equations

- Consider the following second order mean-field SDE:

$$\mathrm{d}\dot{X}_t = (b * \mu_t)(X_t, \dot{X}_t)\mathrm{d}t + \mathrm{d}L_t^{(\alpha)}, \qquad \text{(M-SDE)}$$

  where μ_t is the time marginal distribution of $(X_t, \dot{X}_t)$.

- Suppose that $f = f(t, x, v)$ is the density of the time marginal distribution of $(X_t, \dot{X}_t)$. By Itô's formula, f solves the following kinetic nonlinear Fokker-Planck equation:

$$\partial_t f + v \cdot \nabla_x f = \Delta_v^{\frac{\alpha}{2}} f - \mathrm{div}_v((b * f) f). \qquad \text{(NFPE)}$$

Nonlinear Fokker-Planck equations

- If $b(x, v) = b(v)$ and then $V_t := \dot{X}_t$ solves the following first order mean-field SDE:

$$\mathrm{d}V_t = (b * \mu_t)(V_t)\mathrm{d}t + \mathrm{d}L_t^{(\alpha)},$$

  where μ_t is the time marginal distribution of V_t.
- The density of V_t solves the following non-degenerate nonlinear Fokker-Planck equation:

$$\partial_t \rho = \Delta^{\frac{\alpha}{2}} \rho - \mathrm{div}((b * \rho)\rho).$$

Motivation-examples

- (Vlasov-Poisson-Fokker-Planck equation)
  $d = 3$; $b = b(x) = x/|x|^{d-2}$.
- (Vorticity form of Navier-Stokes equation)
  $d = 2, 3$; $b = b(v)$: Biot-Savart law.
- (Surface quasi-geostropic equation)
  $d = 2$; $b = b(v) = (-v_2/|v|^3, v_1/|v|^3)$: Riesz tranform.
- (Fractional porous medium equation with viscosity)
  $b = b(v) = v/|v|^{d-s}$ with $s \in (0, d)$.

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- Aim 2: Strong and weak well-posedness of second (first) order mean-field SDE (M-SDE);
- Aim 3: Smoothness and long time behavior of the solution $f(t,x,v)$.

Anisotropic scaling

- Consider the following simple second order SDE:

$$\mathrm{d}X_t = V_t \mathrm{d}t, \quad \mathrm{d}V_t = \mathrm{d}L_t^{(\alpha)}.$$

- We have the following scaling:

$$(X_t, V_t) = \left(\int_0^t L_s^{(\alpha)} \mathrm{d}s, L_t^{(\alpha)} \right) \sim (t^{\frac{1+\alpha}{\alpha}} X_1, t^{\frac{1}{\alpha}} V_1).$$

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- For $\boldsymbol{p} = (p_x, p_v)$ and $\boldsymbol{a} = (1+\alpha, 1)$, we introduce the anisotropic distance

$$|(x, v)|_{\boldsymbol{a}} := |x|^{\frac{1}{1+\alpha}} + |v|$$

and mixed-$L^{\boldsymbol{p}}$ norm

$$\|f\|_{L^{\boldsymbol{p}}} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x, v)|^{p_x} \mathrm{d}x \right)^{p_v/p_x} \mathrm{d}v \right)^{1/p_v}.$$

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- Define

$$\boldsymbol{a} \cdot \frac{1}{\boldsymbol{p}} := \frac{1+\alpha}{p_x} + \frac{1}{p_v}.$$

Anisotropic Besov space

- For $r > 0$ and $z \in \mathbb{R}^{2d}$, we also introduce the ball

$$B_r^{\boldsymbol{a}} := \{z' \in \mathbb{R}^{2d} : |z|_{\boldsymbol{a}} \leqslant r\}.$$

- Let χ_0^a be a symmetric C^∞-function on $\mathbb{R}^{2d}$ with

$$\chi_0^{\boldsymbol{a}}(\xi) = 1 \text{ for } \xi \in B_1^{\boldsymbol{a}} \text{ and } \chi_0^{\boldsymbol{a}}(\xi) = 0 \text{ for } \xi \notin B_2^{\boldsymbol{a}}.$$

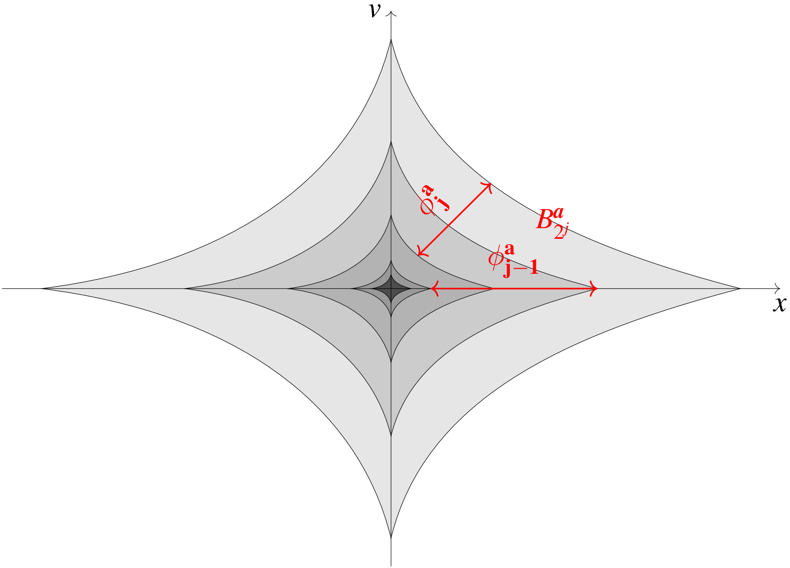
- For $j \in \mathbb{N}$, we define

$$\phi_j^{\boldsymbol{a}}(\xi) := \begin{cases} \chi_0^{\boldsymbol{a}}(2^{-j\boldsymbol{a}}\xi) - \chi_0^{\boldsymbol{a}}(2^{-(j-1)a}\xi), & j \geqslant 1, \\ \chi_0^{\boldsymbol{a}}(\xi), & j = 0, \end{cases}$$

where for $s \in \mathbb{R}$ and $\xi = (\xi_1, \xi_2)$,

$$2^{s\boldsymbol{a}}\xi = (2^{s(1+\alpha)}\xi_1, 2^s\xi_2).$$

- $\sum_{j \geqslant 0} \phi_j^{\boldsymbol{a}}(\xi) = 1.$



- Let $\mathcal{S}$ be the space of all Schwartz functions on $\mathbb{R}^{2d}$ and $\mathcal{S}'$ the dual space of $\mathcal{S}$, called the tempered distribution space.
- For given $j \geqslant 0$, the dyadic block operator $\mathcal{R}_j^a$ is defined on $\mathcal{S}'$ by

$$\mathcal{R}_j^{\boldsymbol{a}} f(z) := (\phi_j^{\boldsymbol{a}} \hat{f})\check{}(z) = \check{\phi}_j^{\boldsymbol{a}} * f(z),$$

where the convolution is understood in the distributional sense.

Definition 1 (Anisotropic Besov spaces)

Let $s \in \mathbb{R}$ and $\boldsymbol{p} \in [1, \infty]^2$. The anisotropic Besov space is defined by

$$\mathbf{B}^s_{\boldsymbol{p},q;\boldsymbol{a}} := \left\{ f \in \mathcal{S}' : \|f\|_{\mathbf{B}^s_{\boldsymbol{p},q;\boldsymbol{a}}} := \left(\sum_{j \geqslant 0} \left(2^{jsq} \|\mathcal{R}^{\boldsymbol{a}}_j f\|^q_{\boldsymbol{p}} \right) \right)^{1/q} < \infty \right\}.$$

Similarly, for any $p \in [1, \infty]$, one defines the isotropic Besov spaces $\mathbf{B}^s_{p,q}$ in $\mathbb{R}^d$.

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Similarly, for any $p \in [1,\infty]$, one defines the isotropic Besov spaces $\mathbf{B}^s_{p,q}$ in $\mathbb{R}^d$.

- Set $\mathbf{B}^s_{\boldsymbol{p};a} := \mathbf{B}^s_{\boldsymbol{p},\infty;a}$ and $\mathbf{B}^s_p := \mathbf{B}^s_{p,\infty}$.
- $\mathbf{B}^{s_0}_{p,1} \subset \mathbf{B}^{s_0}_{p,q} \subset \mathbf{B}^{s_1}_{p,q} \subset \mathbf{B}^{s_1}_{p,\infty}, \quad s_1 \leq s_0.$
- $\mathbf{B}^s_{\infty;\boldsymbol{a}} = \mathbf{B}^{\frac{s}{1+\alpha}}_{\infty;x} \cap \mathbf{B}^s_{\infty;v} = \mathbf{C}^{\frac{s}{1+\alpha}}_x \cap \mathbf{C}^s_v, \quad s > 0$ and $s \notin \mathbb{N}$.

Examples

- Any finite measure μ in $\mathbb{R}^d$ belongs to $\mathbf{B}_1^0$.
- For given $\gamma \in (0, d)$, let $K(x) = |x|^{-\gamma}$, $x \in \mathbb{R}^d$. Then for any $p \in (\frac{d}{\gamma}, \infty]$, it holds that

$$K \in \mathbf{B}_p^{d/p-\gamma}.$$

- Suppose $K \in \mathbf{B}_p^s$ for some $s \in \mathbb{R}$ and $p \in [1, \infty]$. Let

$$K_1(x, v) = K(x),\ K_2(x, v) = K(v),\ \boldsymbol{p}_1 = (p, \infty),\ \boldsymbol{p}_2 = (\infty, p).$$

Then we have

$$\|K_1\|_{\mathbf{B}^{(1+\alpha)s}_{\boldsymbol{p}_1,a}} \asymp \|K\|_{\mathbf{B}^s_p} \asymp \|K_2\|_{\mathbf{B}^s_{\boldsymbol{p}_2;\boldsymbol{a}}}.$$

Well-posedness of SDE with singular drifts

- Let $b \in L_t^{q_b}\mathbf{B}_{p_b}^{\beta_b}$ and consider the following SDE:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}L_t^{(\alpha)}. \tag{SDE}$$

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- Let $\varepsilon \in (0,1)$ and

$$b^\varepsilon(t,x) := \varepsilon^{\alpha-1} b(\varepsilon^\alpha t, \varepsilon x), \quad L_t^{(\alpha),\varepsilon} := \varepsilon^{-1} L_{\varepsilon^\alpha t}^{(\alpha)}, \quad X_t^\varepsilon := \varepsilon^{-1} X_{\varepsilon^\alpha t}.$$

Then

$$\mathrm{d}X_t^\varepsilon = b^\varepsilon(t, X_t^\varepsilon)\mathrm{d}t + \mathrm{d}L_t^{(\alpha),\varepsilon}$$

and

$$\|b^\varepsilon\|_{L_t^{q_b}\mathbf{B}_{p_b}^{\beta_b}} \sim \varepsilon^{\alpha-1+\beta_b-\frac{d}{p_b}-\frac{\alpha}{q_b}} \|b\|_{L_t^{q_b}\mathbf{B}_{p_b}^{\beta_b}}.$$

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- In the sense of this scaling, the sub-critical condition is:

$$\boxed{-\beta_b + \frac{d}{p_b} + \frac{\alpha}{q_b} < \alpha - 1} \tag{A}$$

Well-known results

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- (Priola 2010, Zhang 2012, Xie-Zhang 2020, Ling-Zhao 2021, Athreya-Butkovsky-Mytnik 2020, Chen-Zhang-Zhao 2021,...)
  $\alpha \in (1, 2)$ and $\alpha \in (0, 1)$.
- (Chaudru de Raynal 2017, H.-Wu-Zhang 2020, Chaudru de Raynal-Menozzi 2022...)
  Second order case.

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  $\to \beta_b \in [-1, 0)$ and div$b = 0$ (H.-Zhang 2023)
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$$\beta_b \leq 0 + \text{(A)} \Rightarrow \text{weak well-posedness;}$$

$$-\beta_b + \frac{d}{p_d} + \frac{\alpha}{q_b} < \frac{3}{2}\alpha - 2 \Rightarrow \text{strong well-posedness.}$$

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- $\beta_b > -\frac{1}{2}$ is dropped.
- It can not cover some singular kernels like $b(t,x) = x/|x|^d \in \mathbf{B}_{p_b}^{\beta_b}$ with

$$-\beta_b + \frac{d}{p_b} = d - 1(> \alpha - 1).$$

Initial data and singular kernels

- Consider the following Mean-field SDE ($b \in L_t^{q_b}\mathbf{B}_{p_b}^{\beta_b}$):

$$\mathrm{d}X_t = (b * \mu_t)(X_t)\mathrm{d}t + \mathrm{d}L_t^{(\alpha)}.$$

- Assume $\mu_t \sim \mu_0 \in \mathbf{B}_{p_0}^{\beta_0}$ for some $\beta_0 \leq 0$. Then,

$$b * \mu \in L_t^{q_b}\mathbf{B}_{p,x}^{\beta}, \quad \text{with} \begin{cases} \beta = \beta_b + \beta_0 \\ 1 + 1/p = 1/p_b + 1/p_0. \end{cases}$$

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- Sub-critical condition:

$$\boxed{-\beta_0 + \frac{d}{p_0} - \beta_b + \frac{d}{p_b} + \frac{\alpha}{q_b} < d + \alpha - 1} \qquad \text{(B)}$$

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- $(\beta_0, p_0) = (0, 1) \to$ Chaudru de Raynal-Jabir-Menozzi 2022.

Examples

- Let $1 \le s < d$.

$$b(x) = \nabla |x|^{1-s} \in \dot{\mathbf{B}}_{p_b}^{-\beta_b} \quad \text{with} \quad -\beta_b + \frac{d}{p_b} = s.$$

- Condition (B) is the following

$$-\beta_0 + \frac{d}{p_0} < d - s + \alpha - 1.$$

- Let $\theta \in [0, \alpha - 1)$.

$$b(x) = |\nabla|^{\theta} \delta \in \mathbf{B}_1^{-\theta}.$$

- Condition (B) is the following

$$-\beta_0 + \frac{d}{p_0} < \alpha - 1 - \theta.$$

Main results

Kinetic nonlinear FPE

(H) Let $\alpha \in (1,2]$, $q_b \in (\frac{\alpha}{\alpha-1}]$ and $\boldsymbol{p}_0, \boldsymbol{p}_b \in [1,\infty]^2$ with $\mathbf{1} \leq 1/\boldsymbol{p}_0 + 1/\boldsymbol{p}_b$.
Let $\beta_0 \in (-\alpha + \frac{\alpha}{q_b}, 0)$.
Assume that

$$-\beta_0 + \boldsymbol{a} \cdot \frac{d}{\boldsymbol{p}_0} - \beta_b + \boldsymbol{a} \cdot \frac{d}{\boldsymbol{p}_b} + \frac{\alpha}{q_b} \leq (\alpha + 2)d + \alpha - 1$$

and

$$-2\beta_0 + \boldsymbol{a} \cdot \frac{d}{\boldsymbol{p}_0} - \beta_b + \boldsymbol{a} \cdot \frac{d}{\boldsymbol{p}_b} + \frac{\alpha}{q_b} < (\alpha + 2)d + \alpha.$$

- Set

$$\kappa_0 := \|f_0\|_{\mathbf{B}^{\beta_0}_{\boldsymbol{p}_0,a}}, \quad \kappa_b := \|b\|_{L^{q_b}_t \mathbf{B}^{\beta_b}_{\boldsymbol{p}_b,a}}.$$

Kinetic nonlinear FPE

Theorem 2

Suppose **(H)** holds and $\kappa_b < \infty$. There is a constant $C_0 > 0$ such that if

$$\kappa_0 \kappa_b \leq C_0, \tag{1}$$

then there is a unique smooth solution f to (NFPE).

Kinetic nonlinear FPE

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then there is a unique smooth solution f to (NFPE). Moreover,

(i) if $\boldsymbol{a} \cdot d/\boldsymbol{p}_b > \alpha - \alpha/q_b - 1$ and f_0 is a probability density function. Then for any $\beta \geq 0$

$$\sup_{t \geq 1} \left(t^{\frac{(\alpha+2)d - \boldsymbol{a} \cdot (d/\boldsymbol{p}_0)}{\alpha}} \|f(t)\|_{\mathbf{B}^{\beta}_{\boldsymbol{p}_0;\boldsymbol{a}}} \right) < \infty.$$

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(ii) we can drop the smallness assumption (1) if $\boldsymbol{p}_0 = (1,1)$ or $\mathrm{div} b \equiv 0$, $\boldsymbol{p}_0 = (p_0, p_0)$ and $f_0 \in \cup_{q \in [1,\infty)} \mathbf{B}^{\beta_0}_{\boldsymbol{p}_0,q;\boldsymbol{a}}$.

Second order mean-field SDE

- Consider the following second order mean-field SDE:

$$\mathrm{d}\dot{X}_t = (b * \mu_t)(X_t, \dot{X}_t)\mathrm{d}t + \mathrm{d}L_t^{(\alpha)}, \tag{2}$$

where μ_t is the time marginal distribution of $(X_t, \dot{X}_t)$ and assume $(X_0, \dot{X}_0) \sim f_0$.

Theorem 3

Under the same conditions as in Theorem 2,

- if

$$-\beta_0 + \boldsymbol{a} \cdot \frac{d}{\boldsymbol{p}_0} - \beta_b + \boldsymbol{a} \cdot \frac{d}{\boldsymbol{p}_b} + \frac{\alpha}{q_b} < (\alpha + 2)d + \alpha - 1,$$

then there is a unique weak solution to (2);

- If

$$-\beta_0 + \boldsymbol{a} \cdot \frac{d}{p_0} - \beta_b + \boldsymbol{a} \cdot \frac{d}{p_b} + \frac{\alpha}{q_b} < (\alpha + 2)d + \frac{3}{2}\alpha - 2$$

and $(1 - \Delta_x)^{\frac{2+\alpha}{4(1+\alpha)}} b \in L^{q_b}\mathbf{B}^{\beta_b}_{\boldsymbol{p}_b;\boldsymbol{a}}$, then there is a unique strong solution to (2).

Applications

Fractional Vlasov-Poisson-Fokker-Planck equation

- Let $d \geq 3$. Consider the following fractional Vlasov-Poisson-Fokker-Planck equation:

$$\partial_t f + v \cdot \nabla_x f = \Delta_v^{\frac{\alpha}{2}} f + \gamma \nabla U \cdot \nabla_v f, \tag{VPFP}$$

where $\alpha \in (1,2]$, $\gamma = \pm 1$ stands for the attractive or repulsive force in physics, respectively, and

$$U(t,x) := \int_{\mathbb{R}^{2d}} \frac{f(t,x-y,v)}{|y|^{d-2}} \mathrm{d}y \mathrm{d}v.$$

- Two cases for the well-posedness:

$$f_0 \in \mathbf{B}^{0}_{((1+\alpha)d/(2\alpha),1),a} \; (\alpha = 2 \text{ Carrillo-Soler 1997}) \quad \text{and} \quad f_0 \in \mathbf{B}^{-(2\alpha+1)/2}_{(\infty,1),a}.$$

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- Decay estimate for the force:

$$\|\nabla U(t)\|_\infty \lesssim t^{-\frac{(1+\alpha)(d-1)}{\alpha}}, \quad t \geq 1.$$

When $\alpha = 2$ and $\|f_0\|_{L^1}$ is small enough, it is obtained in Ono-Strauss 2000 (Here we only require $\|f_0\|_{\mathbf{B}^{\beta_0}_{p_0,a}}$ small enough. There is no any restriction on $\|f_0\|_{L^1}$).

- Well-posedness for the related mean-field SDE, which provides a microscopic probabilistic explanation.

Fractional Navier-Stocks equation

- Consider the following 2-dim and 3-dim vorticity fractional Navier-Stocks equation in $\mathbb{R}^2$ ($\mathbb{R}^3$):

$$\partial_t \omega = \Delta^{\frac{\alpha}{2}} \omega + u \cdot \nabla \omega, \quad d = 2; \tag{2D}$$

$$\partial_t \omega = \Delta^{\frac{\alpha}{2}} \omega + u \cdot \nabla \omega + \omega \cdot \nabla u, \quad d = 3. \tag{3D}$$

- Velocity u can be recovered from ω by the Biot-Savart law: $u = K_d * \omega$, $d = 2, 3$, where

$$K_2(x) = \frac{1}{2\pi}\left(\frac{-x_2}{|x|^2}, \frac{-x_1}{|x|^2}\right), \quad K_3(x)h = \frac{1}{4\pi}\frac{x \times h}{|x|^3}, \ h \in \mathbb{R}^3.$$

2D case

(Well-posedness)

- When $p_0 = 1$:

$$\omega_0 \in \cup_{q\in[1,\infty)} \mathbf{B}^{2-\alpha}_{1,q}, \quad \omega_0 \in L^1 \ \ (\alpha = 2 \ \text{Ben-Artzi 1994});$$

$$\omega_0 \in \mathbf{B}^{2-\alpha}_{1,\infty} (\text{small enough}).$$

▷ It seems open to give a well-posedness result for any $\omega_0 \in \mathbf{B}^0_{1,\infty}$.

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  - It seems open to give a well-posedness result for any $\omega_0 \in \mathbf{B}^0_{1,\infty}$.
- When $p_0 = \infty$:

$$\omega_0 \in \mathbf{B}^{-\frac{1+\alpha}{2}+}_{\infty}.$$

  - In this case, we can apply it to the two dimensional Brownian white noise initial data $\omega_0 \in \mathbf{B}^{-1-}_{\infty}$.
- Well-posedness for the related mean-field SDE.

Well-known results

- Giga, Miyakawa, Osada (1988) established the existence of 2d Navier-Stokes flow with measures as initial vorticity, the uniqueness only for atomic part of the initial measure being small.
- Gallagher and Gallay (2005) solved the uniqueness problem for the 2d Navier-Stokes equation with a measure as initial vorticity.
- Zhang (2021) obtain the existence of weak solutions for mean-field SDE with $b \in L^q L^p$ by the maximal principle, where $\frac{d}{p} + \frac{2}{q} < 2$.

- Giga,Y.,Miyakawa,T.,Osada,H.: Two-dimensional Navier-Stokes flow with measures as initial vorticity. *Arch. Rational Mech. Anal.* 104, 223-250 (1988).
- Gallagher I. and Gallay T. : Uniqueness for the two-dimensional Navier-Stokes equation with a measure as initial vorticity *Math. Ann.* 332, 287-327 (2005).
- Zhang X. : Weak solutions of McKean-Vlasov SDEs with supercritical drifts. To appear in *Commun. Math. Stat.* Preprint version available at https://arxiv.org/abs/2010.15330 (2021).

3D case

- (Well-posedness) $\omega_0 \in \mathbf{B}^0_{3/\alpha}$ and $\omega_0 \in \mathbf{B}_\infty^{-(\alpha+1)/2+}$ small enough.

3D case

- (Well-posedness) $\omega_0 \in \mathbf{B}^0_{3/\alpha}$ and $\omega_0 \in \mathbf{B}^{-(\alpha+1)/2+}_{\infty}$ small enough.
- (Decay) for any $p \in [2, \infty]$ and $\beta \geq 0$:

$$\|\omega(t)\|_{\mathbf{B}^{\beta}_{p}} \lesssim t^{-\frac{3-3/p}{\alpha}}, \quad t \geq 1.$$

- Well-posedness for the related mean-field SDE, which provides a microscopic probabilistic explanation.

Thank you for attention!

Merci beaucoup!