SDEs with supercritical distribution drifts

Zimo Hao

Bielefeld University

Joint work with Xicheng Zhang

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Overview

- Consider the following SDE

$$dX_t = b(t, X_t)dt + \sqrt{2}dW_t, \qquad (1)$$

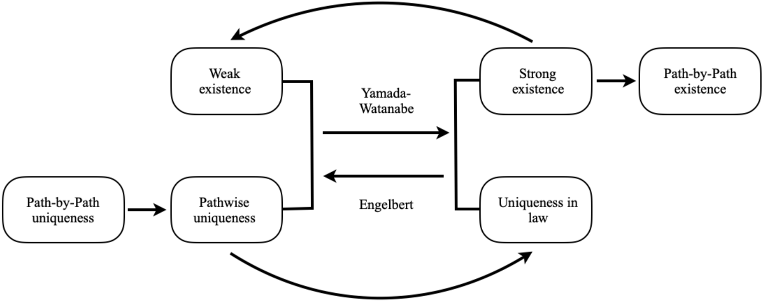
where $(W_t)_{t\geq 0}$ is a standard d-dimensional Brownian motion and $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a measurable function.

  - Weak solution: $(\Omega, \mathscr{F}, \mathbf{P}, (\mathscr{F}_s)_{s\geq 0}, W, X)$;
  - Strong solution: $(\Omega, \mathscr{F}, \mathbf{P}, (\mathscr{F}_s)_{s\geq 0}, W) \Rightarrow X = \Phi(X_0, W)$;
  - Maringale solution: $\mathbb{P} \in \mathscr{P}(\mathcal{C}_T)$, for all $f \in \mathbf{C}^2(\mathbb{R}^d)$

$$f(\omega_t) - f(\omega_0) - \int_0^t (\Delta + b \cdot \nabla) f(\omega_s) ds \quad \text{is a } \mathbb{P}\text{-martingale;}$$

  - Path-by-path solution: for any path $t \to W_t(\omega)$, the solution solves the ODE (1).
  - Uniqueness in law; Pathwise uniqueness; Path-by-path uniqueness.

- Regularization by noise.



- ▷ (Stroock-Varadhan): Weak solution $\iff$ Martingale solution;
- ▷ (Barlow): Uniqueness in law $\not\Rightarrow$ Existence of strong solution.
- ▷ (Shaposhnikov-Wresch, Anzeletti): Many counterexamples.

- ▷ Yamada, T. and Watanabe, S. (1971). On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.*
- ▷ Engelbert, H. J. (1991). On the theorem of T. Yamada and S. Watanabe. *Stochastics Stochastics Rep.*
- ▷ Stroock, D. W. and Varadhan, S. S. R. Multidimensional diffusion processes, volume 233 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. *Springer-Verlag, Berlin*, 1979.
- ▷ Barlow, M. T. (1982). One-dimensional stochastic differential equations with no strong solution. *J. London Math. Soc.*
- ▷ Shaposhnikov, A. and Wresch, L. (2022). Pathwise vs. path-by-path uniqueness. *Ann. Inst. Henri Poincaré Probab. Stat.*
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What can we say if b is not a function?

- Brox diffusion (white noise); Other noises.
- $b = \nabla U$ with some Hölder potential;
- (Weak solution): $X_t = X_0 + A_t^b + W_t$, where

$$A_t^b := \lim_{n\to\infty} \int_0^t b_n(s, X_s) \mathrm{d}s \quad \text{exists.}$$

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- (Martingale solution):
  - For any $f \in \mathbf{C}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, consider the related BKE

    $$\partial_t u + \Delta u + b \cdot \nabla u + f = 0, \quad u(T) = 0.$$

    We call $\mathbb{P} \in \mathscr{P}(\mathcal{C}_T)$ a martingale solution if

    $$u(t, \omega_t) - u(t, \omega_0) - \int_0^t f(r, \omega_r) \mathrm{d}r \quad \text{is a } \mathbb{P}\text{-martingale.}$$

  - N. Ethier and G. Kurtz. Markov Processes: Characterization and Convergence. *Wiley series in probability and mathematical statistic. Wiley*, 1986.

Scale analysis

- Let $\dot{\mathbf{H}}_p^\alpha$ be the homogenous Bessel potential space, where $\alpha \leq 0$ and $p \in [1, \infty]$ and suppose for some $q \in [1, \infty]$

$$b \in L^q(\mathbb{R}_+; \dot{\mathbf{H}}_p^\alpha),$$

and SDE (1) admits a solution denoted by X. For $\lambda > 0$, we define

$$X_t^\lambda := \lambda^{-1} X_{\lambda^2 t}, \quad W_t^\lambda := \lambda^{-1} W_{\lambda^2 t}, \quad b^\lambda(t,x) := \lambda b(\lambda^2 t, \lambda x).$$

- Then we have

$$\mathrm{d}X_t^\lambda = b^\lambda(t, X_t^\lambda)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t^\lambda,$$

where

$$\|b^\lambda\|_{L^q(\mathbb{R}_+;\dot{\mathbf{H}}_p^\alpha)} = \lambda^{1+\alpha-\frac{d}{p}-\frac{2}{q}} \|b\|_{L^q(\mathbb{R}_+;\dot{\mathbf{H}}_p^\alpha)}.$$

- As $\lambda \to 0$,

Subcritical: $\frac{d}{p} + \frac{2}{q} < 1 + \alpha$;

Critical: $\frac{d}{p} + \frac{2}{q} = 1 + \alpha$;

Supercritical: $\frac{d}{p} + \frac{2}{q} > 1 + \alpha$.

A well-defined restriction on α

- Consider the related PDE:

$$\partial_t u = \Delta u + b \cdot \nabla u + f.$$

- Assume $b \in \mathbf{C}^\alpha$ with the differentiability index $\alpha < 0$.
- By the Schauder theory, u is at most in $\mathbf{C}^{2+\alpha}$.
- To make the product $b \cdot \nabla u$ meaningful, we need to stipulate that $1 + 2\alpha > 0$, which implies $\alpha > -\frac{1}{2}$.
  - (Delarue-Diel 2016) rough path & (Cannizzaro-Chouk 2018) paracontrolled calculus: $b \in \mathbf{C}^{-2/3+}$ is some Gaussian noise.
  - (Question) Arbitrary function b? $\alpha \to -1$?

Well-known results

SEU: Strong existence-uniqueness; WEU: Weak existence-uniqueness;
WE: Weak existence; EUP: Existence-uniqueness of path-by-path solution;
EUE: Existence-uniqueness of energy solution.

Value of α	Subcritical	Critical	Supercritical
$\alpha = 0$	SEU: $\mathrm{V}^{79}_{[1]}$, $\mathrm{KR}^{05}_{[2]}$, $\mathrm{Z}^{05,10}_{[3,4]}$ EUP: $\mathrm{D}^{07}_{[5]}$, $\mathrm{ALL}^{23}_{[6]}$	WEU&SEU: $\mathrm{BFGM}^{19}_{[7]}$, $\mathrm{K}^{21}_{[8]}$, $\mathrm{RZ}^{21}_{[9]}$, $\mathrm{KM}^{23}_{[10]}$	WE: $\mathrm{ZZ}^{21}_{[11]}$
$\alpha \in [-\frac{1}{2}, 0)$	WEU: $\mathrm{BC}^{01}_{[12]}$, $\mathrm{FIR}^{17}_{[13]}$, $\mathrm{ZZ}^{17}_{[14]}$	–	–
$\alpha \in [-1, -\frac{1}{2})$	EUE : $\mathrm{GP}^{23}_{[15]}$	–	–

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[5] A. M. Davie. *Int. Math. Res. Not. IMRN* **24**. [6] L. Anzeletti, K. Lê and C. Ling. arXiv:2304.06802.
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Weak well-posedness of subcritical SDEs with $\alpha \in (-1, -\frac{1}{2})$

Main results

($\mathbf{H}^{\rm sub}$) Let $(\alpha, p, q) \in (-1, -\frac{1}{2}] \times [2, \infty)^2$ with $\frac{d}{p} + \frac{2}{q} < 1 + \alpha$. Suppose that

$$\kappa_1^b := \|b\|_{\mathbb{L}^q_T \mathbf{B}^{\alpha}_{p,q}} < \infty \quad \text{and} \quad \kappa_2^b := \|\mathrm{div} b\|_{\mathbb{L}^q_T \mathbf{B}^{-2-\alpha}_{p,q/(q-1)}} < \infty.$$

Theorem 1 (H.-Zhang 2023)

*Under the condition (**$H^{\rm sub}$**), there is a unique weak solution to SDE* (1). *Moreover, $t \to A_t^b$ has finite p-variation with some $p < 2$.*

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Theorem 1 (H.-Zhang 2023)

Under the condition ($\boldsymbol{H}^{\rm sub}$), there is a unique weak solution to SDE (1). *Moreover, $t \to A_t^b$ has finite p-variation with some $p < 2$.*

- Suppose that $b \in \mathbb{L}_T^q \mathbf{B}_{p,1}^{-1/2}$ with $\frac{d}{p} + \frac{2}{q} < \frac{1}{2}$. Then ($\mathbf{H}^{\rm sub}$) holds for $\alpha = -\frac{1}{2}$. Moreover, when $\mathrm{div} b = 0$, ($\mathbf{H}^{\rm sub}$) holds.
- For any Lipschitz function $g : \mathbb{R}^d \to \mathbb{R}$,

$$\int_0^t g(X_s) \mathrm{d}A_s^b \quad \text{is a Young integral.}$$

Example:Gaussian noises

- For given $\gamma \in (d-2, d)$, we define the Gaussian noise b by the following covariance

$$\mathbb{E}b(f)b(g) = \int_{\mathbb{R}^d} \hat{f}(\xi)\hat{g}(-\xi)|\xi|^{-\gamma}\Big(\mathbb{I}_{d\times d} - \frac{\xi \otimes \xi}{|\xi|^2}\Big)\mathrm{d}\xi.$$

- Then we have for almost surely ω

$$b(\omega, \cdot) \in \cap_{p\in[1,\infty)} \mathbf{B}^{-1+}_{p,loc}(\mathbb{R}^d) \quad \mathrm{div} b(\omega) = 0.$$

Weak solutions to supercritical SDEs with $\alpha = -1$

The setting

- We assume $d \geq 2$, $b \in L_T^q \mathbf{H}_p^{-1}$ with $p, q \in [2, \infty]$,

$$\frac{d}{p} + \frac{2}{q} < 1, \quad \mathrm{div} b = 0.$$

The setting

- We assume $d \geq 2$, $b \in L_T^q \mathbf{H}_p^{-1}$ with $p, q \in [2, \infty]$,

$$\frac{d}{p} + \frac{2}{q} < 1, \quad \text{div} b = 0.$$

- Let $b_n \in \mathbf{C}_b^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ with $\lim_{n\to\infty} \|b_n - b\|_{L_T^q \mathbf{H}_p^{-1}} = 0$ and consider the following approximating SDE

$$X_t^n = X_0 + \int_0^t b_n(s, X_s^n) \mathrm{d}s + \sqrt{2} W_t.$$

- We denote the distribution of $(X_t^n)_{t\in[0,T]}$ by $\mathbb{P}_n \in \mathscr{P}(C([0,T]; \mathbb{R}^d))$.

Main results

Theorem 2 (H.-Zhang 2023)

i) For any $\mathscr{F}_0$ measurable random variable X_0, $\{\mathbb{P}_n\}_{n=1}^{\infty}$ is **tight** *in $\mathscr{P}(C([0,T];\mathbb{R}^d))$.*

ii) Moreover, if the distribution of X_0 has an L^2 density w.r.t. the Lebesgue measure, then there is a continuous process $(X_t)_{t\in[0,T]}$ such that

$$X_t = X_0 + \lim_{n\to\infty}\int_0^t b_n(r,X_r)\mathrm{d}r + \sqrt{2}W_t,$$

where the limit here is taken in $L^2(\Omega)$.

iii) Let $\mathbb{P}$ be the law of the solution $(X_t)_{t\in[0,T]}$. The following almost surely Markov property holds: there is a Lebesgue zero set $\mathscr{N}\subset(0,T)$ such that for all $s\in[0,T)\backslash\mathscr{N}$

$$\mathbb{E}_{\mathbb{P}}[f(\omega_t)|\mathscr{B}_s] = \mathbb{E}_{\mathbb{P}}[f(\omega_t)|\omega_s], \quad 0\le s\le t\le T,\ f\in\mathbf{C}_b(\mathbb{R}^d).$$

iv)When $b\in L^2([0,T]\times\mathbb{R}^d)$ or $b\in L_T^\infty\mathbf{B}_{\infty,2}^{-1}$ (critical & ill-defined), there is only one accumulation point of $\{\mathbb{P}_n\}_{n=1}^{\infty}$. That is for any $b_n\to b$, $\mathbb{P}_n$ converges to the distribution of $(X_t)_{t\in[0,T]}$.

Example: Particle system with singular kernels

- Consider the following singular interaction particle system in $\mathbb{R}^{Nd}$:

$$\mathrm{d}X_t^{N,i} = \sum_{j\neq i} \gamma_j K(X_t^{N,i} - X_t^{N,j})\mathrm{d}t + \sqrt{2}\mathrm{d}W_t^{N,i}, \quad i = 1, \cdots, N, \tag{2}$$

  where $K \in \mathbf{H}_\infty^{-1}(\mathbb{R}^d; \mathbb{R}^d)$ is divergence free, $W_t^{N,i}, i = 1, \cdots, N$ are N-independent standard d-dimensional Brownian motions, $\gamma_j \in \mathbb{R}$ and initial value has an L^2-density.
- (Jabin-Wang 2018) Existence of the related FPE and propagation of chaos. (The existence of a solution to the SDE (2) appears to be open).
- As a result, we have the weak existence to the N-particle system SDE (2).

Example: GFF and super-diffusive

- Let $d = 2$, $\varepsilon \in (0,1]$ and b_ε be a Gaussian field with
$$\mathbb{E}b_\varepsilon(f)b_\varepsilon(g) = \int_{|\xi|\le 1/\varepsilon} \hat{f}(\xi)\hat{g}(-\xi)\Big(\mathbb{I}_{d\times d} - \frac{\xi\otimes\xi}{|\xi|^2}\Big)\mathrm{d}\xi.$$
- When $\varepsilon \to 0$, $b := \lim_\varepsilon b_\varepsilon$ formally satisfies
$$b := \nabla^\perp \xi := (-\partial_{x_2}\xi_1, \partial_{x_1}\xi_2) \in \mathbf{C}^{-1-} \quad \mathrm{div} b = 0,$$
where $\xi = \xi(x)$ is the two-dimensional Gaussian Free Field (GFF)
- (Super-diffusive)
When $\varepsilon = 1$, $\mathbb{E}|X_t|^2 \asymp t\sqrt{\ln t}$
(Cannizzaro-HaunschmidSibitz-Toninelli 2022)
(Chatzigeorgiou-Morfe-Otto-Wang 2022).
- For any $p \in (2,\infty)$
$$\sup_{\varepsilon<1/2} \Big\|\frac{b_\varepsilon}{\sqrt{\ln(1/\varepsilon)}}\Big\|_{\mathbf{H}^{-1}_{p,loc}} < \infty, \quad a.s.$$
By our results, one sees that the solutions $\{X^\varepsilon_t\}_{[0,T]}$ to the following approximation SDEs is tight
$$\mathrm{d}X^\varepsilon_t = \frac{b_\varepsilon(X^\varepsilon_t)}{\sqrt{\ln(1/\varepsilon)}}\mathrm{d}t + \sqrt{2}\mathrm{d}W_t.$$

Example: GFF and super-diffusive

- (Yang-Yang 2024):

$$\mathrm{d}X_t^\varepsilon = \frac{b_\varepsilon(X_t^\varepsilon)}{\sqrt{\ln(1/\varepsilon)}}\mathrm{d}t + \sqrt{2}\mathrm{d}W_t.$$

- X_t^ε convergences to a Brownian motion as $\varepsilon \to 0$.
- Imply the assumption $\alpha \geq -1$ is sharp.

Thank you!