Singular kinetic equations

Zimo Hao[1],
郝子墨

zhao@math.uni-bielefeld.de

(Joint work with Xicheng Zhang[2], Rongchan Zhu[2] and Xiangchan Zhu[3])

[1]Wuhan University

[2]Beijing Institute of Technology

[3]Chinese Academy of Sciences

arXiv:2108.05042

28 August, 2022

Table of contents

Motivation

- Consider the following N-particles system:

$$\begin{cases} \mathrm{d}X_t^{N,i} = V_t^{N,i}\mathrm{d}t, \\ \mathrm{d}V_t^{N,i} = b(Z_t^{N,i})\mathrm{d}t + \frac{1}{N}\sum_{j\neq i} K(X_t^{N,i} - X_t^{N,j})\mathrm{d}t + \sqrt{2}\mathrm{d}B_t^i, \end{cases}$$

where $i = 1, 2, .., N$

  - $Z^{N,i} = (X^{N,i}, V^{N,i}) \in \mathbb{R}^{2d}$: position and velocity of particle number i
  - b: the force field depending on both position and velocity
  - K: interaction kernel.
  - B^i: independent Brownian motions (random phenomena)

DDSDE

- When b and K are smooth, the solution of the N-particles system $Z^{N,i}$ convergences to the solution to the following Distribution Dependent SDE (DDSDE):

$$\begin{cases} \mathrm{d}X_t = V_t \mathrm{d}t \\ \mathrm{d}V_t = b(Z_t)\mathrm{d}t + \int_{\mathbb{R}^d} K(X_t - y)\mu_t(\mathrm{d}y)\mathrm{d}t + \mathrm{d}B_t, \end{cases} \tag{1}$$

where μ_t is the distribution of X_t and B_t is a standard BM.

DDSDE

- When b and K are smooth, the solution of the N-particles system $Z^{N,i}$ convergences to the solution to the following Distribution Dependent SDE (DDSDE):

$$\begin{cases} \mathrm{d}X_t = V_t \mathrm{d}t \\ \mathrm{d}V_t = b(Z_t)\mathrm{d}t + \int_{\mathbb{R}^d} K(X_t - y)\mu_t(\mathrm{d}y)\mathrm{d}t + \mathrm{d}B_t, \end{cases} \tag{1}$$

where μ_t is the distribution of X_t and B_t is a standard BM.

- Well-known results

Jabin and Wang (JFA-16) Assume $K \in L^\infty$. Well-posedness for Fokker-Plank equation and propagation of chaos.

Chaudru de Raynal (AIHP-15) Assume $K \equiv 0$ and $b \in \mathbf{C}_x^\alpha \cap \mathbf{C}_v^\beta$ with $\alpha > \frac{2}{3}$, $\beta > 0$. Well-posedness for SDE equation (1).

Zhang (China Math-18) Assume $K \equiv 0$ and $(1-\Delta_x)^{\frac{1}{3}} b \in L^p$ with $p > 2(2d+1)$. Well-posedness for SDE equation (1).

Wang and Zhang (SIAM-16) Hölder Dini assumption

Chaudru de Raynal, Honoré and Menozii 18 Chain case

Chen and Zhang (JMPA-18) & Hao, Wu and Zhang (JMPA-20) Non-local case

Jair 19 & Lacker 21 Propagation and chaos

. . . .

Motivation

- Recently, the study of the SDEs driven by singular drift b is more and more popular, especially the distribution case. Such singular diffusions appear as models for stochastic processes in random media, i.e. b would also be random, but independent of BM B.

Motivation

- Recently, the study of the SDEs driven by singular drift b is more and more popular, especially the distribution case. Such singular diffusions appear as models for stochastic processes in random media, i.e. b would also be random, but independent of BM B.
- For the following first order SDE:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \mathrm{d}B_t,$$

  - Brox (AoP-86): Brox diffusion (b is a space white noise)
  - Delarue and Diel (PTRF-16): 1-d distribution drift b (Rough path)
  - Cannizzaro and Chouk (AoP-18): Multidimensional case with distribution drift b (Paracontrolled calculus).

Motivation

- Recently, the study of the SDEs driven by singular drift b is more and more popular, especially the distribution case. Such singular diffusions appear as models for stochastic processes in random media, i.e. b would also be random, but independent of BM B.
- For the following first order SDE:

$$dX_t = b(X_t)dt + dB_t,$$

  - Brox (AoP-86): Brox diffusion (b is a space white noise)
  - Delarue and Diel (PTRF-16): 1-d distribution drift b (Rough path)
  - Cannizzaro and Chouk (AoP-18): Multidimensional case with distribution drift b (Paracontrolled calculus).
- (Question:)
  Could we understand the kinetic DDSDE (1) with singular noise b?

Our model

- Let b be a Gaussian noise in $\mathbb{R}^{2d}$ with spectral measure μ, i.e. for any $f, g \in \mathscr{S}(\mathbb{R}^{2d})$,

$$\mathbb{E}[\langle b,f\rangle\langle b,g\rangle] = \int_{\mathbb{R}^{2d}} \hat{f}(\zeta)\hat{g}(-\zeta)\mu(\mathrm{d}\zeta).$$

Our model

- Let b be a Gaussian noise in $\mathbb{R}^{2d}$ with spectral measure μ, i.e. for any $f, g \in \mathscr{S}(\mathbb{R}^{2d})$,

$$\mathbb{E}[\langle b,f\rangle\langle b,g\rangle] = \int_{\mathbb{R}^{2d}} \hat{f}(\zeta)\hat{g}(-\zeta)\mu(\mathrm{d}\zeta).$$

  - For example, when $\mu(\mathrm{d}\zeta) = \mathrm{d}\zeta$ is the Lebsgue measure of $\mathbb{R}^{2d}$, b is a white noise in $\mathbb{R}^{2d}$; when $\mu(\mathrm{d}\zeta) = \delta(\mathrm{d}\xi)\mathrm{d}\eta$ with $\zeta = (\xi, \eta) \in \mathbb{R}^{2d}$, $b(x, v) = b(v)$ is the white noise in $\mathbb{R}^d_v$.
  - When b is a white noise, $b \in \mathscr{C}^{-\frac{1}{2}-}$ is not a function, but only a distribution.

Our model

- Consider the following kinetic DDSDE with distribution drift b and interaction kernel K,

$$\begin{cases} \mathrm{d}X_t = V_t \mathrm{d}t \\ \mathrm{d}V_t = b(Z_t)\mathrm{d}t + \int_{\mathbb{R}^d} K(X_t - y)\mu_t(\mathrm{d}y)\mathrm{d}t + \mathrm{d}B_t. \end{cases} \qquad (\star)$$

Our model

- Consider the following kinetic DDSDE with distribution drift b and interaction kernel K,

$$\begin{cases} \mathrm{d}X_t = V_t \mathrm{d}t \\ \mathrm{d}V_t = b(Z_t)\mathrm{d}t + \int_{\mathbb{R}^d} K(X_t - y)\mu_t(\mathrm{d}y)\mathrm{d}t + \mathrm{d}B_t. \end{cases} \quad (\star)$$

- (Questions:)
  - Definition-Nonlinear martingale problem
    -Paracontrolled calculus
  - Existence and Uniqueness-what's the condition of μ and K.
  - Propagation of chaos

Kinetic operator

- To set up the martingale problem to DDSDE $(\star)$, it is important to establish the well-posedness for the related PDE:

$$\partial_t u = \Delta_v u + v \cdot \nabla_x u + b \cdot \nabla u + f, \quad u(0) = \varphi, \qquad \text{(SKE)}$$

  where $u = u(t,z) = u(t,x,v)$.
- The following operator is called kinetic operator

$$\mathscr{L} := \partial_t - \Delta_v - v \cdot \nabla_x.$$

Kinetic operator

- In 1934, Kolmogorov found the fundamental solution of $\mathscr{L}$ is the density $p_t(x,v)$ of the process $\left(\sqrt{2}\int_0^t B_s \mathrm{d}s, \sqrt{2}B_t\right)$ and

$$p_t(x,v) = \Big(\frac{2\pi t^4}{3}\Big)^{-\frac{d}{2}} \exp\Big(-\frac{3|x|^2 + |3x - 2tv|^2}{4t^3}\Big).$$

Kinetic operator

- In 1934, Kolmogorov found the fundamental solution of $\mathscr{L}$ is the density $p_t(x,v)$ of the process $\left(\sqrt{2}\int_0^t B_s \mathrm{d}s, \sqrt{2}B_t\right)$ and

$$p_t(x,v) = \Big(\frac{2\pi t^4}{3}\Big)^{-\frac{d}{2}} \exp\Big(-\frac{3|x|^2 + |3x-2tv|^2}{4t^3}\Big).$$

- More precisely, for any $z=(x,v)$, denote by the semigroup

$$P_t f(z) := \Gamma_t p_t * \Gamma_t f(z) = \mathbb{E} f\Big(x + tv + \sqrt{2}\int_0^t B_s \mathrm{d}s, v + B_t\Big),$$

where

$$\Gamma_t f(z) := f(\Gamma_t z) := f(x+tv, v).$$

Then,

$$u(t) := P_t \varphi + \int_0^t P_{t-s} f(s) \mathrm{d}s$$

solves the following kinetic equation:

$$\mathscr{L} u = f, \quad u(0) = \varphi.$$

Scaling

- Notice that

$$p_t(x,v) = t^{-2d} p_1(t^{\frac{3}{2}} x, t^{\frac{1}{2}} v).$$

- The scaling of x and v is $3:1$ in the kinetic equations. So, it is nature to consider the following metric:

$$|z_1 - z_2|_a := |x_1 - x_2|^{\frac{1}{3}} + |v_1 - v_2|, \quad z_i := (x_i, v_i), i = 1, 2.$$

Scaling

- Notice that

$$p_t(x,v) = t^{-2d} p_1(t^{\frac{3}{2}} x, t^{\frac{1}{2}} v).$$

- The scaling of x and v is $3:1$ in the kinetic equations. So, it is nature to consider the following metric:

$$|z_1 - z_2|_a := |x_1 - x_2|^{\frac{1}{3}} + |v_1 - v_2|, \quad z_i := (x_i, v_i), i = 1, 2.$$

- For any $\alpha > 0$, define the anistrophic Hölder-Zygmund space $\mathbf{C}_a^\alpha$ with the following norm

$$\|f\|_{\mathbf{C}_a^\alpha} := \|f\|_\infty + \sup_{h \neq 0} \frac{\|\delta_h^{[\alpha]+1} f\|_\infty}{|h|_a},$$

where $[]$ is the Gauss function and $\delta_h f(z) := f(z+h) - f(z)$.

Anistropic Besov space

- For $\alpha < 0$, we need Besov space to extend the definition of anistropic Hölder-Zygmund space $\mathbf{C}_a^\alpha$.

Anistropic Besov space

- For $\alpha < 0$, we need Besov space to extend the definition of anistropic Hölder-Zygmund space $\mathbf{C}_a^\alpha$.
- Let
$$B_j := \{\zeta \in \mathbb{R}^{2d}, 2^{j-1} < |\zeta|_a < 2^{j+1}/3\}$$
for $j \geq 0$ and $B_{-1} := \{|\zeta| < 2/3\}$.
- By a technical construction, there is a unity of partition $\{\phi_j^a\}_{j\geq -1} \subset \mathbf{C}_0^\infty$ belonging to $\cup_{j\geq -1} B_j$ and

$$\phi_j^a(\xi, \eta) = \phi_0^a(2^{-3j}\xi, 2^{-j}\eta).$$

Anistrophic Besov space

- For given $j \geq -1$, the block operator $\mathcal{R}_j^a$ is defined on $\mathscr{S}'$ by

$$\mathcal{R}_j^a f(z) := \mathscr{F}^{-1}(\phi_j^a \mathscr{F}(f))(z) = \mathscr{F}^{-1}(\phi_j^a) * f(z).$$

- For any $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$, define the **anistrophic Besov** space $\mathbf{B}_{p,q}^{a,\alpha}$ with the following norm:

$$\|f\|_{\mathbf{B}_{p,q}^{\alpha,a}} := \Big(\sum_{i \geq -1} 2^{\alpha q j} \|\mathcal{R}_j^a f\|_p^q \Big)^{\frac{1}{q}}.$$

- It is well-known that for $\alpha > 0$, $\mathbf{C}_a^\alpha = \mathbf{B}_{\infty,\infty}^{a,\alpha}$. For simplicity, we denote by

$$\mathbf{C}_a^\alpha := \mathbf{B}_{\infty,\infty}^{a,\alpha}$$

for all $\alpha \in \mathbb{R}$ in the sequel.

Weighted anistrophic Besov space

- The noise has a blow up when the position-velocity space $\mathbb{R}^{2d}$ is not compact. We need a weight to balance the increasing (blow up) as $z \to \infty$ in $\mathbb{R}^{2d}$
- Let $\mathscr{P}_W$ be the set of all function in the following form

$$\rho_\delta(z) = ((1+|x|^2)^{1/3} + |v|^2)^{-\delta/2} \asymp (1+|z|_a)^{-\delta}, \quad \delta \in \mathbb{R}.$$

- For any $\alpha \in \mathbb{R}$ and $\rho \in \mathscr{P}_W$, define the **weight anistrophic Besov space** $\mathbf{C}_a^\alpha(\rho)$ with the following norm:

$$\|f\|_{\mathbf{C}_a^\alpha(\rho)} := \|\rho f\|_{\mathbf{C}_a^\alpha}.$$

Kinetic Hölder space

- Recall
$$\Gamma_t f(z) := f(\Gamma_t z), \quad \Gamma_t z := (x + tv, v).$$
- Let $\alpha \in (0,2)$, $\rho \in \mathscr{P}_W$ and $T > 0$. Define
$$\mathbb{S}^{\alpha}_{T,a}(\rho) := \Big\{ f : \|f\|_{\mathbb{S}^{\alpha}_{T,a}(\rho)} := \|f\|_{L^\infty_T \mathbf{C}^{\alpha}_a(\rho)} + \|f\|_{\mathbf{C}^{\alpha/2}_{T;\Gamma} L^\infty(\rho)} < \infty \Big\},$$
where for $\beta \in (0,1)$,
$$\|f\|_{\mathbf{C}^{\beta}_{T;\Gamma} L^\infty(\rho)} := \sup_{0 \leqslant t \leqslant T} \|f(t)\|_{L^\infty(\rho)} + \sup_{0 < |t-s| \leqslant 1} \frac{\|f(t) - \Gamma_{t-s} f(s)\|_{L^\infty(\rho)}}{|t-s|^\beta}.$$

Kinetic Hölder space

- Recall
$$\Gamma_t f(z) := f(\Gamma_t z), \ \ \Gamma_t z := (x + tv, v).$$
- Let $\alpha \in (0, 2)$, $\rho \in \mathscr{P}_W$ and $T > 0$. Define
$$\mathbb{S}^\alpha_{T,a}(\rho) := \Big\{ f : \|f\|_{\mathbb{S}^\alpha_{T,a}(\rho)} := \|f\|_{L^\infty_T \mathbf{C}^\alpha_a(\rho)} + \|f\|_{\mathbf{C}^{\alpha/2}_{T;\Gamma} L^\infty(\rho)} < \infty \Big\},$$
where for $\beta \in (0, 1)$,
$$\|f\|_{\mathbf{C}^\beta_{T;\Gamma} L^\infty(\rho)} := \sup_{0 \leqslant t \leqslant T} \|f(t)\|_{L^\infty(\rho)} + \sup_{0 < |t-s| \leqslant 1} \frac{\|f(t) - \Gamma_{t-s} f(s)\|_{L^\infty(\rho)}}{|t-s|^\beta}.$$
- The reason why we introduce this kinetic Hölder space is because
$$\|P_t f - \Gamma_t f\|_\infty \le C_\delta t^{\frac{\delta}{2}} \|f\|_{\mathbf{C}^\delta_a} \quad \delta \in (0, 2).$$

Schauder's estimate

- Denote by

$$\mathscr{I} := \int_0^t P_{t-s}\, \mathrm{d}s.$$

Theorem 1 (Schauder's estimate)

Let $\beta \in (0,2)$ and $\rho \in \mathscr{P}_W$. For any $T > 0$, there is a constant $C = C(d,\beta,\rho,T) > 0$ such that for all $f \in L^\infty_T \mathbf{C}_a^{-\beta}(\rho)$,

$$\|\mathscr{I} f\|_{\mathbb{S}^{2-\beta}_{T,a}(\rho)} \leq C \|f\|_{L^\infty_T \mathbf{C}_a^{-\beta}(\rho)}.$$

Linear singular kinetic equation

- Consider the following kinetic equation

$$\partial_t u = \Delta_v u + v \cdot \nabla_x u + b \cdot \nabla_v u + f, \quad u(0) = \varphi.$$

- We are interested in the following noise b

$$b \in \mathbf{C}_a^{-\alpha}(\rho_\kappa), \quad \alpha \in (\frac{1}{2}, \frac{2}{3}) \text{ and } \kappa \in (0, 1).$$

- By Schauder's estimate, the best regularity of the solution u is in

$$L^\infty([0, T]; \mathbf{C}_a^{2-\alpha}(\rho_\kappa)).$$

Difficulties

1. (ill-defined problem) $b \cdot \nabla_v u$ does not make sense since

$$\mathbf{C}_a^{-\alpha} \times \mathbf{C}_a^{1-\alpha} \ni (b, \nabla_v u) \to b \cdot \nabla_v u \in \mathbf{C}_a^{\alpha \wedge (1-\alpha)} \text{ only if } 1 - 2\alpha > 0.$$

Difficulties

1. (ill-defined problem) $b \cdot \nabla_v u$ does not make sense since

$$\mathbf{C}_a^{-\alpha} \times \mathbf{C}_a^{1-\alpha} \ni (b, \nabla_v u) \to b \cdot \nabla_v u \in \mathbf{C}_a^{\alpha \wedge (1-\alpha)} \text{ only if } 1 - 2\alpha > 0.$$

2. (weight lose problem) The weight ρ_κ here always make technical difficult, roughly speaking,

$$u \in \mathbf{C}_a^{\cdots}(\rho_\kappa) + b \in \mathbf{C}_a^{\cdots}(\rho_\kappa) \Rightarrow b \cdot \nabla_v u \in \mathbf{C}_a^{\cdots}(\rho_{2\kappa}).$$

By Schauder's estimate again,

$$u \in \mathbf{C}_a^{\cdots}(\rho_{2\kappa}) \Rightarrow b \cdot \nabla_v u \in \mathbf{C}_a^{\cdots}(\rho_{3\kappa}) \Rightarrow u \in \mathbf{C}_a^{\cdots}(\rho_{3\kappa}),$$

$$\Rightarrow u \in \mathbf{C}_a^{\cdots}(\rho_{\infty\kappa}) \quad \text{blow up!}$$

Methods to solve the difficulties

1. (Paracontrolled calculus) Transfer the ill-defined problem from

$$b \cdot \nabla_v u \quad \text{to} \quad b \cdot \nabla_v \mathscr{I} b.$$

$$\text{from} \quad \int_0^t f(w_s) \mathrm{d}w_s \quad \text{to} \quad \int_0^t w_s \mathrm{d}w_s \text{ similar in rough path.}$$

More precisely, let $u = u(b, b \cdot \nabla_v \mathscr{I} b)$.

- This method was first proposed by Gubinelli, Imkeller and Perkowski in 2015 for $\partial_t - \Delta$.
- However, our model is $\partial_t - \Delta_v - v \cdot \nabla_x$. This kinetic operator is not a multiplication operator like $\partial_t - \Delta$.
- It is interesting and not easy to establish the paracontrolled calculus for the kinetic equations.

Methods to solve the difficulties

1. (Paracontrolled calculus) Transfer the ill-defined problem from

$$b \cdot \nabla_v u \quad \text{to} \quad b \cdot \nabla_v \mathscr{I} b.$$

$$\text{from} \quad \int_0^t f(w_s) \mathrm{d} w_s \quad \text{to} \quad \int_0^t w_s \mathrm{d} w_s \ \text{ similar in rough path.}$$

More precisely, let $u = u(b, b \cdot \nabla_v \mathscr{I} b)$.

- This method was first proposed by Gubinelli, Imkeller and Perkowski in 2015 for $\partial_t - \Delta$.
- However, our model is $\partial_t - \Delta_v - v \cdot \nabla_x$. This kinetic operator is not a multiplication operator like $\partial_t - \Delta$.
- It is interesting and not easy to establish the paracontrolled calculus for the kinetic equations.

2. (Localization)

- This method is in Zhang, Zhu and Zhu's paper: arXiv:2007.06783 for $\partial_t - \Delta$.
- Key points:

$$\|f\|_{\mathbf{C}^\beta(\rho)} \asymp \sup_z \rho(z) \|f \chi_z\|_{\mathbf{C}^\beta}$$

and

$$|\nabla_v \rho(z)| \lesssim \rho_1(z) \rho(z).$$

Renormalization of Gauss noise b

- Recall that

$$\mathbb{E}[\langle b,f\rangle\langle b,g\rangle] = \int_{\mathbb{R}^{2d}} \hat{f}(\zeta)\hat{g}(-\zeta)\mu(\mathrm{d}\zeta).$$

- Suppose that for $\zeta = (\xi,\eta) \in \mathbb{R}^{2d}$

$$\mu(\mathrm{d}\xi,\mathrm{d}\eta) = \mu(-\mathrm{d}\xi,\mathrm{d}\eta) = \mu(\mathrm{d}\xi,-\mathrm{d}\eta).$$

($\mathbf{H}^\beta$) For some $\beta \in (\frac{1}{2},\frac{2}{3})$,

$$\sup_{\zeta'\in\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \frac{\mu(\mathrm{d}\zeta)}{(1+|\zeta+\zeta'|_a)^{2\beta}} < \infty.$$

Main result for linear kinetic equations

Theorem 2
Let $\alpha \in (\frac{1}{2}, \frac{2}{3})$, $\vartheta := \frac{9}{2-3\alpha}$, $\kappa_f \in \mathbb{R}$,

$$\delta := (2\vartheta + 2)\kappa \leqslant 1 \quad \text{and} \quad \bar{\kappa} := (2\vartheta + 1)\kappa + \kappa_f.$$

Assume that $(\mathbf{H}^\beta)$ holds for some $\beta < \alpha$. For any $T > 0$, $\kappa > 0$,

$$f \in L^\infty_T \mathbf{C}_a^{\alpha-1+}(\rho_{\kappa_f}) \quad \text{or} \quad f = b, \quad \text{and} \quad \varphi \in \mathbf{C}_a^{\alpha+1+}(\rho_{\kappa_f-\kappa}),$$

there is a unique paracontrolled solution $u \in \mathbb{S}_{T,a}^{2-\alpha}(\rho_{\bar{\kappa}})$ to PDE (SKE).

Examples

1. For any $\lambda \in (d - \frac{1}{3}\alpha, d)$,

$$\mu(\mathrm{d}\zeta) = \frac{\mathrm{d}\xi}{|\xi|^\lambda}\delta(\mathrm{d}\eta), \quad \zeta = (\xi, \zeta),$$

where $\mathrm{d}\xi$ is the Lebsgue measure on $\mathbb{R}^d$ and $\delta(\mathrm{d}\eta)$ is the Dirac measure.
When $d = 1$,

$$b(x, v) = b(x) = \partial_x B_H(x)$$

is the derivative of a fractional BM with Hurst parameter $H = \frac{1+\lambda}{2}$.

Examples

1. For any $\lambda \in (d - \frac{1}{3}\alpha, d)$, $\mu(\mathrm{d}\zeta) = \frac{\mathrm{d}\xi}{|\xi|^\lambda}\delta(\mathrm{d}\eta)$.

2. For any $\lambda \in (d - 2\alpha, d)$ and $\lambda \geq 0$,

$$\mu(\mathrm{d}\zeta) = \delta(\mathrm{d}\xi)\frac{\mathrm{d}\eta}{|\eta|^\lambda}, \quad \zeta = (\xi, \zeta).$$

   When $d = 1$, $\lambda = 0$ is admissible, this time $b(x, v) = \xi(v)$ is a space white noise on $\mathbb{R}_v$.

Examples

1. For any $\lambda \in (d - \frac{1}{3}\alpha, d)$, $\mu(\mathrm{d}\zeta) = \frac{\mathrm{d}\xi}{|\xi|^\lambda}\delta(\mathrm{d}\eta)$.
2. For any $\lambda \in (d - 2\alpha, d)$ and $\lambda \geq 0$, $\mu(\mathrm{d}\zeta) = \delta(\mathrm{d}\xi)\frac{\mathrm{d}\eta}{|\eta|^\lambda}$.
3. For any $\lambda_1, \lambda_2 \in [0, d)$ with $3\lambda_1 + \lambda_2 > 4d - 2\alpha$,

$$\mu(\mathrm{d}\zeta) = \frac{\mathrm{d}\xi\mathrm{d}\eta}{|\xi|_1^\lambda |\eta|_2^\lambda}, \quad \zeta = (\xi, \zeta).$$

Notice that, when $d = 1$, $\lambda_2 = 0$ is admissible.
In this case, b can be regarded as the white noise in v-part.

Singular kinetic DDSDE

- Now, let's go back to the original problem:

$$\begin{cases} \mathrm{d}X_t = V_t \mathrm{d}t \\ \mathrm{d}V_t = b(Z_t)\mathrm{d}t + \int_{\mathbb{R}^d} K(X_t - y)\mu_t(\mathrm{d}y)\mathrm{d}t + \mathrm{d}B_t, \end{cases} \quad (\star)$$

where $b \in \mathbf{C}_a^{-\alpha}(\rho_\kappa)$ for some $\alpha \in (\frac{1}{2}, \frac{2}{3})$, $\kappa > 0$ and satisfying the condition **(H**$^\beta$**)** with some $\beta < \alpha$.

Singular kinetic DDSDE

- Now, let's go back to the original problem:

$$\begin{cases} \mathrm{d}X_t = V_t \mathrm{d}t \\ \mathrm{d}V_t = b(Z_t)\mathrm{d}t + \int_{\mathbb{R}^d} K(X_t - y)\mu_t(\mathrm{d}y)\mathrm{d}t + \mathrm{d}B_t, \end{cases} \qquad (\star)$$

  where $b \in \mathbf{C}_a^{-\alpha}(\rho_\kappa)$ for some $\alpha \in (\frac{1}{2}, \frac{2}{3})$, $\kappa > 0$ and satisfying the condition $(\mathbf{H}^\beta)$ with some $\beta < \alpha$.
- Let $\mathcal{P}(E)$ be the set of all probability measures over some Banach space E. Denote by $\mathcal{P}_\delta(\mathbb{R}^{2d})$ be the space of all $\nu \in \mathcal{P}(\mathbb{R}^{2d})$ with

$$\int_{\mathbb{R}^{2d}} \rho_{-\delta}(z)\nu(\mathrm{d}z) \asymp \int_{\mathbb{R}^{2d}} (1 + |z|_a)^\delta \nu(\mathrm{d}z) < \infty, \quad \delta \geq 0.$$

Singular kinetic DDSDE

- Now, let's go back to the original problem:

$$\begin{cases} \mathrm{d}X_t = V_t \mathrm{d}t \\ \mathrm{d}V_t = b(Z_t)\mathrm{d}t + \int_{\mathbb{R}^d} K(X_t - y)\mu_t(\mathrm{d}y)\mathrm{d}t + \mathrm{d}B_t, \end{cases} \qquad (\star)$$

  where $b \in \mathbf{C}_a^{-\alpha}(\rho_\kappa)$ for some $\alpha \in (\frac{1}{2}, \frac{2}{3})$, $\kappa > 0$ and satisfying the condition $(\mathbf{H}^\beta)$ with some $\beta < \alpha$.

- Let $\mathcal{P}(E)$ be the set of all probability measures over some Banach space E. Denote by $\mathcal{P}_\delta(\mathbb{R}^{2d})$ be the space of all $\nu \in \mathcal{P}(\mathbb{R}^{2d})$ with

$$\int_{\mathbb{R}^{2d}} \rho_{-\delta}(z)\nu(\mathrm{d}z) \asymp \int_{\mathbb{R}^{2d}} (1 + |z|_a)^\delta \nu(\mathrm{d}z) < \infty, \quad \delta \geq 0.$$

- For any $T > 0$, denote by $\mathcal{C}_T := C([0,T]; \mathbb{R}^{2d})$ and let $\mathscr{B}_t$ be the natural σ-filtration, and z be the canonical process over $\mathcal{C}_T$, i.e.

$$z_t(\omega) = (x_t(\omega), v_t(\omega)) = \omega_t, \quad \omega \in \mathcal{C}_T.$$

Nonlinear martingale problem

- For any continuous curve $\mu : [0,T] \to \mathcal{P}(\mathbb{R}^d)$ with respect to the weak convergence, define

$$\mathscr{L}_t^{\mu} := \Delta_v + v \cdot \nabla_x + (b + K * \mu_t) \cdot \nabla_v.$$

- Let $f \in L_T^\infty \mathbf{C}_b(\mathbb{R}^{2d})$, $\varphi \in \mathbf{C}_a^{\alpha+1+}(\mathbb{R}^{2d})$ and $\vartheta = \frac{9}{2-3\alpha}$. We call $(u, f, \varphi) \in A_T^{\mu}$ if $u \in \mathbb{S}_{T;a}^{2-\alpha}(\rho_{2(\vartheta+1)\kappa})$ is the paracontrolled solution to the following linear kinetic equation:

$$\partial_t u + \mathscr{L}_t^{\mu} u = f, \quad u(T) = \varphi.$$

Nonlinear martingale problem

Definition 3 (Nonlinear martingale problem)

Let $\delta > 0$. A probability measure $\mathbb{P} \in \mathcal{P}(\mathcal{C}_T)$ is called a nonlinear martingale solution to DDSDE ($\star$) starting from $\nu \in \mathcal{P}_\delta(\mathbb{R}^{2d})$, if $\mathbb{P} \circ z_0^{-1} = \nu$ and for all $(u, f, \varphi) \in A_T^\mu$,

$$M_t := u(t, z_t) - u(0, z_0) - \int_0^t f(s, z_s)\mathrm{d}s$$

is a $\mathbb{P}$-martingale respect to $(\mathscr{B}_t)_{t\in[0,T]}$, where $\mu_t = \mathbb{P} \circ x_t^{-1}$.

Nonlinear martingale problem

Definition 3 (Nonlinear martingale problem)

Let $\delta > 0$. A probability measure $\mathbb{P} \in \mathcal{P}(\mathcal{C}_T)$ is called a nonlinear martingale solution to DDSDE $(\star)$ starting from $\nu \in \mathcal{P}_\delta(\mathbb{R}^{2d})$, if $\mathbb{P} \circ z_0^{-1} = \nu$ and for all $(u, f, \varphi) \in A_T^{\mu}$,

$$M_t := u(t, z_t) - u(0, z_0) - \int_0^t f(s, z_s)\mathrm{d}s$$

is a $\mathbb{P}$-martingale respect to $(\mathscr{B}_t)_{t\in[0,T]}$, where $\mu_t = \mathbb{P} \circ x_t^{-1}$.

Theorem 4 (Main result)

Assume that $(\mathbf{H}^{\beta})$ and $K \in \mathbf{C}^{\frac{\alpha-1+}{3}}$. For any $\delta > 0$, there exists at least one nonlinear martingale solution to DDSDE $(\star)$. Moreover, if K is bounded measurable, then there exists at most one solution.

N-particle system with singular drift

- Recall the following N-particles system:

$$\begin{cases} \mathrm{d}X_t^{N,i} = V_t^{N,i}\mathrm{d}t, \\ \mathrm{d}V_t^{N,i} = b(Z_t^{N,i})\mathrm{d}t + \frac{1}{N}\sum_{j\neq i} K(X_t^{N,i} - X_t^{N,j})\mathrm{d}t + \sqrt{2}\mathrm{d}B_t^i. \end{cases} \tag{$\mathcal{N}$}$$

- Denote by $B(\mathbf{z}) := B(z_1, .., z_N) := (b(z_1), ..., b(z_N))$. Then,

$$B(\mathbf{z}) \cdot \nabla_{\mathbf{v}} \mathscr{I}_N B(\mathbf{z}) = (b \cdot \nabla_{v_1} \mathscr{I} b(z_1), ..., b \cdot \nabla_{v_1} \mathscr{I} b(z_N))$$

  is well-defined, where $\mathscr{I}_N := \mathscr{I}^{\otimes N}$.
- Hence, by the same argument above, we have the well-posedness for the linear martingale problem to N-particle system ($\mathcal{N}$).

Propagation of chaos

- Denote by $\mu^N \in \mathcal{P}(\mathcal{C}_T)$ and $\mu \in \mathcal{P}(\mathcal{C}_T)$ be the distribution of $Z^{N,i}$ and the distribution to the solution of non-linear martingale problem respectively.
- Let b_n be the modifier of b.
- Denote by $\mu_n^N \in \mathcal{P}(\mathcal{C}_T)$ and $\mu_n \in \mathcal{P}(\mathcal{C}_T)$ be the distribution of the i-th particle in the N-particle system $(\mathcal{N})$ with $b = b_n$ and the distribution to the solution of DDSDE with $b = b_n$ respectively.

Propagation of chaos

- Denote by $\mu^N \in \mathcal{P}(\mathcal{C}_T)$ and $\mu \in \mathcal{P}(\mathcal{C}_T)$ be the distribution of $Z^{N,i}$ and the distribution to the solution of non-linear martingale problem respectively.
- Let b_n be the modifier of b.
- Denote by $\mu_n^N \in \mathcal{P}(\mathcal{C}_T)$ and $\mu_n \in \mathcal{P}(\mathcal{C}_T)$ be the distribution of the i-th particle in the N-particle system ($\mathcal{N}$) with $b = b_n$ and the distribution to the solution of DDSDE with $b = b_n$ respectively.
- (Lacker 2021)

$$\|\mu_n^N - \mu_n\|_{TV} \leq 4\sqrt{T}(1 + \|K\|_\infty^2)e^{3T\|K\|_\infty^2}\frac{1}{N}.$$

- Notice that the constant

$$C := 4\sqrt{T}(1 + \|K\|_\infty^2)e^{3T\|K\|_\infty^2}$$

is independence of n!

Propagation of chaos

- By the argument in DDSDE, we have for any $N \in \mathbb{N}$,

$$\mu_n^N \to \mu^N, \;\; \mu_n \to \mu \;\text{ weakly as } n \to \infty.$$

- $$\begin{array}{ccc} \mu^N & \xleftarrow{\;w\;} & \mu_n^N \\ w \Big\downarrow & & \Big\downarrow TV \\ \mu & \xleftarrow{\;w\;} & \mu_n \end{array}$$

Propagation of chaos

- By the argument in DDSDE, we have for any $N \in \mathbb{N}$,

$$\mu_n^N \to \mu^N, \quad \mu_n \to \mu \ \text{ weakly as } n \to \infty.$$

-

$$\begin{array}{ccc} \mu^N & \xleftarrow{\ w\ } & \mu_n^N \\ {\scriptstyle w}\big\downarrow & & \big\downarrow{\scriptstyle TV} \\ \mu & \xleftarrow{\ w\ } & \mu_n \end{array}$$

- For precisely, for any $f \in C_b(\mathcal{C}_T)$,

$$\begin{aligned} |\mu(f) - \mu^N(f)| \leq & \lim_{n\to\infty} |\mu(f) - \mu_n(f)| \\ & + \sup_n |\mu_n(f) - \mu_n^N(f)| \\ & + \lim_{n\to\infty} |\mu^N(f) - \mu_n^N(f)| \\ \leq & \frac{C}{N} \|f\|_\infty. \end{aligned}$$

Nonlinear Fokker-Planck equation

- Finally, we'd like to study the density of the martingale solution to DDSDE.
- (Question:)
  1. Whether the solution admit a density?
  2. Whether the density satisfy the following Fokker-Plank equation:

$$\partial_t u = \Delta_v u - v \cdot \nabla u - \mathrm{div}_v\Big((b + K * \langle u \rangle)u\Big), \quad u(0) = u_0,$$

where $\langle u \rangle(t,x) := \int_{\mathbb{R}^d} u(t,x,v)\mathrm{d}v$ and $\nu(\mathrm{d}z) = u_0(z)\mathrm{d}z$?

Assumption: b is the Gauss noise satisfying $(\mathbf{H}^\beta)$ with $\beta < \alpha$ and

$$\mathrm{div}_v b \equiv 0 \quad \text{and} \quad u_0 \in L^1 \cap \mathbf{C}_a^{\alpha+1+}.$$

- Consider the following singular kinetic Fokker-Planck equation

$$\partial_t u + v \cdot \nabla_x = \Delta_v u + b \cdot \nabla_v u + K * \langle u \rangle \cdot \nabla_v u, \quad u(0) = u_0.$$

Here $\langle u \rangle(t,x) := \int_{\mathbb{R}^d} u(t,x,v)\mathrm{d}v$.

Conditions

(H$_1$) $K \in \cup_{\beta>\alpha-1} \mathbf{C}^{\beta/3}$.

(H$_2$) K is bounded, for some $k > 0$,

$$\|\rho_{-k} u_0\|_{L^1} \asymp \int_{\mathbb{R}^{2d}} (1+|z|_a)^k u_0(z)\mathrm{d}z < \infty \quad (\text{energy condition}).$$

and

$$H(u_0) := \int_{\mathbb{R}^{2d}} u_0 \ln u_0 \mathrm{d}z < \infty \quad (\text{entropy condition}).$$

(H$_3$)

$$\int_{\mathbb{R}^{2d}} u_0(z)\mathrm{d}z = 1 \quad (\text{mass condition}).$$

Main result

Theorem 5

Let $\alpha \in (\frac{1}{2}, \frac{2}{3})$ and $(\mathbf{H}^\beta)$ hold for some $\beta < \alpha$.

1. Under the condition (H_1), there is at least a solution $u \in L^\infty_T(\mathbf{C}_a^{2-\alpha}(\rho))$ to nonlinear Fokker-Planck equation with some weight $\rho \in \mathscr{P}_W$.
2. Under the condition (H_2), the solution is unique with
$$\|\rho_{-k}u(t)\|_{L^1} \leq C\|\rho_{-k}u_0\|_{L^1}(\text{energy estimate});$$
$$H(u(t)) \leq H(u_0)(\text{entropy estimate}).$$
3. Under the condition (H_1), (H_2) and (H_3), there is a unique solution u with $u \geq 0$ and
$$\int_{\mathbb{R}^{2d}} u(t,z)\mathrm{d}z = 1(\text{mass conservation}).$$

Moreover, u is the density to the solution of the nonlinear martingale problem with initial data $\nu(\mathrm{d}z) = u_0(z)\mathrm{d}z$.

Thank you!

谢谢！